Hypothesis Testing for Generalized Thurstone Models

Anuran Makur and Japneet Singh

Abstract

In this work, we develop a rigorous hypothesis testing method to determine whether pairwise comparison data is generated by an underlying *generalized Thurstone model* $\mathcal{T}_F$ for a given choice function F. Given n agents, a $\mathcal{T}_F$ model assumes that each agent i has a latent utility parameter w_i and the probability that agent i is preferred over agent j in a pairwise comparison (e.g., a game) is given by $F(w_i - w_j)$. While prior work has predominantly focused on parameter estimation and uncertainty quantification for such models, our work bridges a crucial gap by developing a hypothesis testing approach for $\mathcal{T}_F$ models. We formulate this testing problem in a minimax sense by introducing a notion of separation distance between a general pairwise comparison model and the class of $\mathcal{T}_F$ models. We then derive both upper and lower bounds on the critical threshold of our minimax hypothesis testing problem, which depend on the topology of the underlying observation graph of comparisons. For example, in the setting where all possible pairwise comparisons are observed (i.e., complete observation graph), the critical threshold scales as $\Theta((nk)^{-1/2})$, where k is the number of pairwise comparisons between each pair of agents. Furthermore, we propose a specific hypothesis test inspired by our separation distance for our testing problem, and assess its performance by establishing "time-uniform" upper bounds on type I and type II error probabilities using reverse martingale ideas. To complement this, we also develop a minimax risk lower bound for our testing problem using information-theoretic ideas. Additionally, we prove several auxiliary results over the course of our analysis, such as error bounds on parameter estimation and "time-uniform" confidence intervals. Finally, we conduct several experiments on synthetic and real-world datasets to validate some of our theoretical results and test for $\mathcal{T}_F$ models. In the process, we also propose a data-driven approach to find the threshold of our test.

I. Introduction

Learning rankings from data is a fundamental problem underlying numerous applications, including recommendation systems [1], sports tournaments [2], [3], fine-tuning large language model (LLMs) [4], and social choice theory [5], [6]. The class of generalized Thurstone models (GTMs) [7]–[9], which fall under the broader framework of random utility models, is a widely adopted framework for ranking agents, items, or choices based on given preference data. GTMs include many other models as special cases, most notably the Bradley-Terry-Luce (BTL) model [2], [5], [10], which has been widely studied. Given n agents $[n] = \{1, \ldots, n\}$, GTMs can be construed as likelihood models for pairwise comparisons between pairs of agents. In particular, a GTM $\mathcal{T}_F$ assumes that each agent i is endowed with an unknown utility parameter $w_i \in \mathbb{R}$ and the probability that agent i is preferred over agent j (e.g., i beats j in a game) is given by $F(w_i - w_j)$, where F represents a known choice function which is a cumulative distribution function (CDF).

While GTMs have been utilized in many contexts, e.g., [11], [12], they are parametric models where n utility parameters characterize the model. Indeed, the assumption that pairwise comparison data is governed by a small number of parameters forms the basis of most results on GTMs [3], [13]–[15]. However, such parametric models can sometimes be too restrictive, failing to capture intricacies in real applications [16]–[18]. Notably, GTMs struggle to accommodate context-dependent effects, such as the home-advantage effect observed in sports tournaments [19], [20], where teams may perform differently when playing at home versus away. Furthermore, GTMs assume transitive relationships, which may not hold in real-world datasets. To accurately capture the complex and diverse behaviors observed in real-world data, non-parametric models, e.g., [14], [21], have been studied as an alternative. This conversation raises an important question: *Given pairwise comparison data, can we determine whether it is comes from a specific GTM?* If it does, then we can rely on the vast GTM literature for learning, and if it does not, then we can resort to using other parametric models such as the Mallows model [22] or non-parametric models [23].

Despite extensive research in the area, there is no systematic answer to the above question in the literature, i.e., there is no rigorously analyzed hypothesis test to determine whether given pairwise comparison data conforms to an

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A. Makur is with the Department of Computer Science and the Elmore Family School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, USA (e-mail: amakur@purdue.edu).

J. Singh is with the Elmore Family School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, USA (e-mail: sing1041@purdue.edu).

underlying GTM model. To address this, we study the composite hypothesis testing problem of whether data obeys a GTM $\mathcal{T}_F$ for a given choice function F:

$$\begin{aligned} &H_0 : \mathcal{Z} \sim \mathcal{T}_F \text{ for some choice of } w \in \mathcal{W}, \\ &H_1 : \mathcal{Z} \sim \text{general pairwise comparison model that is not } \mathcal{T}_F, \end{aligned} \tag{1}$$

where $\mathcal{Z}$ denotes the pairwise comparison data, and H_0 and H_1 are the null and alternative hypotheses, respectively and $\mathcal{W}$ denotes the parameter set for weight w.

A. Main Contributions

We analyze the composite hypothesis testing problem outlined in (1) with a specified choice function F. Our main contributions include the following:

1) We frame the hypothesis testing problem in a minimax sense (Section II) by developing a rigorous notion of separation distance to the class of all GTMs that admits tractable analysis (Section III, Theorem 1).
2) We derive upper and lower bounds on the critical threshold for our test (Section III, Theorem 2 and Proposition 2). These bounds exhibit a dependence on the graph induced by the pairwise comparison data (see Section III-B) and are tight for complete graphs.
3) We use the separation distance to propose a hypothesis test and establish various theoretical guarantees for our test. Specifically, we prove "time-uniform" type I and type II error probability upper bounds for our test (Section III, Theorems 4 and 5), and also provide a minimax lower bound.
4) Additionally, we obtain auxiliary results like error bounds on parameter estimation for general pairwise comparison models (Theorem 3) and "time-uniform" confidence intervals under the null hypothesis (Proposition 4).
5) Finally, we validate our theoretical findings through synthetic and real-world experiments, proposing a data-driven approach to determine the test threshold and using the test to determine different choice functions' fit to the data (Section VIII).

B. Related Literature

The class of GTMs has a rich history in the analysis of preference data. Initially proposed by Thurstone [7], these models are widely used in various fields, ranging from psychology [24], economics [25], and more recent applications like aligning LLMs with human preferences [4]. Early foundational works, e.g., [5]–[7], explored different cumulative distribution functions F for modeling choice probabilities, including Gaussian [7], logistic [2], and Laplace [26]. These models and their extensions underlie popular rating systems, such as Elo in chess [11], [12], [27] and TrueSkill in video games [28]. Several recent works have actively explored estimation techniques for Thurstone models. For instance, [13] estimated parameters of Thurstone models when the preference data is derived from general subsets of agents (not specifically pairs), and [14] focused on parameter estimation for GTMs and the effect of graph topology on the estimation accuracy.

Furthermore, a significant portion of the literature has focused on parameter estimation in the special case of the BTL model, e.g., [15], [29]–[33], where two prominent algorithms are spectral ranking [30], [31] and maximum likelihood estimation [27], [34]. Another related line of work is on uncertainty quantification for estimated parameters [35]–[38]. For example, [35] established the asymptotic normality of estimated parameters in the BTL model for both spectral ranking and maximum likelihood estimation, and [36] generalized the asymptotic normality results to a broader class of models such as GTMs and Mallows models.

Despite the extensive work on parameter estimation, relatively few studies have rigorously investigated hypothesis testing for such parametric models. In particular, [39] developed two-sample tests for preference data, [40] studied lower bounds for testing the independence of irrelevant alternatives (IIA) assumption (i.e., BTL and Plackett-Luce models [5], [41]), and [42] developed hypothesis tests for BTL models based on spectral methods. In contrast to these works, we develop hypothesis testing for GTMs using a maximum likelihood framework, complementing the work in [42].

II. Formal Model and Setup

We begin by introducing a general pairwise comparison model that provides a flexible framework encompassing a broad range of established probabilistic models, including the BTL model [2], [5], [10], the Thurstone model [7], and non-parametric models [14], [21]. In this framework, we consider $n \in \mathbb{N}\backslash\{1\}$ agents (or items or choices) $[n]$ engaged in pairwise comparisons. For agents $i, j \in [n]$ with $i \neq j$, let $p_{ij} \in (0,1)$ denote the probability that i is preferred over

j in an "i vs. j" pairwise comparison. This model inherently captures the asymmetric nature of pairwise comparisons, as the outcome of an "i vs. j" comparison may differ from that of a "j vs. i" comparison. This reflects real-world phenomena like "home advantage" that are commonly observed in sports [19], [20]. To model the fact that not all pairwise comparisons may be observed, we assume that we are given an induced observation graph $\mathcal{G} = ([n], \mathcal{E})$, where an edge $(i, j) \in \mathcal{E}$ (with $i \neq j$) exists if and only if comparisons of the form "i vs. j" are observed. Let $E \in \{0, 1\}^{n \times n}$ be the adjacency matrix of $\mathcal{G}$, with $E_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and 0 otherwise. Furthermore, we assume that the edge set $\mathcal{E}$ is symmetric (i.e., $\mathcal{G}$ is undirected), implying that if "i vs. j" comparisons are observed, then "j vs. i" comparisons are observed as well. Additionally, we assume that $\mathcal{G}$ is connected and is fixed a priori (see Proposition 1), independent of the outcomes of observed pairwise comparisons. Also, let $D \in \mathbb{R}^{n \times n}$ the diagonal degree matrix with $D_{ii} = \sum_{j=1}^{n} E_{ij}$ for $i \in [n]$, and $L \triangleq D - E$ be the graph Laplacian matrix. L can be expressed as $L = X^{\mathrm{T}} X$, where $X \in \mathbb{R}^{(|\mathcal{E}|/2) \times n}$ is the matrix formed by collecting row vectors $x_{ij} = e_i - e_j$ for $(i, j) \in \mathcal{E}$ and $j > i$, with e_i being the ith standard basis vector in $\mathbb{R}^n$. For the Laplacian L, we define the semi-norm with respect to L as $\|x\|_L = \sqrt{x^{\mathrm{T}} L x}$, where $x \in \mathbb{R}^n$ and L is the graph Laplacian matrix.

A. Comparison Models

1) Pairwise Comparison Model: We begin by defining a general pairwise comparison model which encompasses a broad range of existing models, such as the BTL model [2], [5], [10], Thurstonian model [7], non-parametric models [14], [21], etc.

Definition 1 (Pairwise Comparison Model)**.** *Given an observation graph $\mathcal{G}$ over the agents $[n]$, we refer to the collection of probability parameters $\{p_{ij} : (i, j) \in \mathcal{E}\}$ as a* pairwise comparison model*.*

Furthermore, we can represent a pairwise comparison model by a *pairwise comparison matrix* $P \in [0, 1]^{n \times n}$ with

$$P_{ij} \triangleq \begin{cases} p_{ij}, & (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We remark that our ensuing analysis can be easily specialized to a symmetric setting where "i vs. j" and "j vs. i" comparisons are equivalent. In this case, E is automatically symmetric as assumed. On the other hand, the symmetry assumption on E is needed in asymmetric settings because GTMs inherently treat "i vs. j" and "j vs. i" comparisons as equivalent, which is not true in general models.

2) GTM model: Next, we describe a GTM for a choice function $F : \mathbb{R} \to [0, 1]$ (a special kind of CDF).

Definition 2 (Generalized Thurstone Model)**.** *Given an observation graph $\mathcal{G}$, a pairwise comparison model is said to be a* generalized Thurstone model *(GTM) $\mathcal{T}_F$ with choice function $F : \mathbb{R} \to [0, 1]$ if there exists a weight (or utility) vector $w \in \mathcal{W}$ such that:*

$$\forall (i, j) \in \mathcal{E}, \;\; p_{ij} = F(w_i - w_j),$$

where $\mathcal{W} \subseteq \mathbb{R}^n$ is a specified convex parameter space (usually $\mathbb{R}^n$ or a compact hypercube in $\mathbb{R}^n$).

The GTM [8], [9] posits that every agent i has a latent utility w_i, and uncertainty in the comparison process is modeled by independent and identically distributed (i.i.d.) noise random variables $X_1, \dots, X_n$ with absolutely continuous CDF $G : \mathbb{R} \to [0, 1]$. The discriminant variables $(w_1 + X_1, \dots, w_n + X_n)$ formed by combining utilities with the noise random variables are then compared to determine the outcomes of pairwise comparisons. Hence, the probability of preferring agent i over j is given by

$$\mathbb{P}(i \text{ preferred over } j) = \mathbb{P}(w_i + X_i > w_j + X_j) = \int_{-\infty}^{\infty} G(y + w_i - w_j) G'(y) \, \mathrm{d}y = F(w_i - w_j). \tag{3}$$

As noted earlier, GTMs encompass a wide range of models as special cases, e.g., Thurstone models [7], BTL models [2], [5], Dawkins models [26], etc. We can also define a pairwise probability matrix $\mathsf{F}(w) \in [0, 1]^{n \times n}$ for a GTM $\mathcal{T}_F$ with weight vector w via

$$(\mathsf{F}(w))_{ij} \triangleq \begin{cases} F(w_i - w_j), & (i, j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We next describe the data generation process for GTMs and general pairwise comparison models alike. For any pair $(i, j) \in \mathcal{E}$, define the outcome of the mth "i vs. j" pairwise comparison between them as the Bernoulli random variable

$$Z_{ij}^m \triangleq \begin{cases} 1, & \text{if } i \text{ preferred over } j \text{ (with probability } p_{ij}), \\ 0, & \text{if } j \text{ preferred over } i \text{ (with probability } 1 - p_{ij}), \end{cases} \tag{5}$$

for $m \in [k_{ij}]$, where k_{ij} denotes the number of observed "i vs. j" comparisons. The given pairwise comparison data is then a collection of these *independent* Bernoulli variables $\mathcal{Z} \triangleq \{Z_{ij}^m : (i,j) \in \mathcal{E},\ m \in [k_{i,j}]\}$. For convenience, we also let $Z_{ij} \triangleq \sum_{m=1}^{k_{ij}} Z_{ij}^m$ and $\hat{p}_{ij} \triangleq Z_{ij}/k_{ij}$.

3) Parameter Estimation for GTM: To present our testing formulation in the sequel, we explain how the parameters of a $\mathcal{T}_F$ model are estimated given pairwise comparison data $\mathcal{Z}$ [14]. First, we define the weighted negative log-likelihood function $l : \mathcal{W} \times [0,1]^{|\mathcal{E}|} \to \mathbb{R}_+ \cup \{+\infty\}$ as

$$l(w; \{\hat{p}_{ij} : (i,j) \in \mathcal{E}\}) \triangleq -\sum_{(i,j)\in\mathcal{E}} \hat{p}_{ij} \log(F(w_i - w_j)) + (1-\hat{p}_{ij}) \log(1 - F(w_i - w_j)). \tag{6}$$

Note that this function represents a weighted variant of the typical log-likelihood function used in parameter estimation [13], [14]. The weights of the $\mathcal{T}_F$ model are estimated by minimizing:

$$\hat{w} \triangleq \operatorname*{arg\,min}_{w \in \mathcal{W}_b} l(w; \{\hat{p}_{ij} : (i,j) \in \mathcal{E}\}), \tag{7}$$

where the constraint set $\mathcal{W}_b \triangleq \{w \in \mathcal{W} : \|w\|_\infty \leq b,\ w^{\mathrm{T}}\mathbf{1} = 0\}$ for some (universal) constant b, $\mathbf{1} \in \mathbb{R}^n$ denotes an all-ones vector, and the constraint $w^{\mathrm{T}}\mathbf{1} = 0$ allows for identifiability of the weights.

B. Assumption on Comparison Models

To facilitate the analysis of the hypothesis testing problem in (1), we introduce a simplifying assumption on the class of general pairwise comparison models. We assume that the pairwise probabilities p_{ij} are bounded away from 0 and 1.

Assumption 1 (Dynamic Range)**.** *There exists a constant $\delta > 0$ such that for any pairwise comparison model under consideration, $p_{ij} \in [\delta, 1-\delta]$ for all $(i,j) \in \mathcal{E}$.*

Note that under the null hypothesis, the Assumption 1 is satisfied by all $\mathcal{T}_F$ models with weights bounded by $F^{-1}(1-\delta)/2$. Subsequently, we assume that the constant b satisfies $b \geq F^{-1}(1-\delta)/2$. For any given pairwise comparison model $\{p_{ij} : (i,j) \in \mathcal{E}\}$, define $w^* \in \mathcal{W}_b$ be the weights of a $\mathcal{T}_F$ model that best approximates this pairwise comparison model in the maximum likelihood sense:

$$w^* \triangleq \operatorname*{arg\,min}_{w \in \mathcal{W}_b} l(w; \{p_{ij} : (i,j) \in \mathcal{E}\}). \tag{8}$$

Finally, we also assume in the sequel that the given choice function F exhibits *strong log-concavity* and has a bounded derivative on $[-2b, 2b]$, i.e., there exists a constant $\alpha, \beta > 0$ such that:

$$\forall x \in [-2b, 2b], \quad -\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log(F(x)) \geq \alpha \quad \text{and} \quad F'(x) \leq \beta. \tag{9}$$

Several popular GTMs including the BTL and Thurstone (Case V) model satisfy both the above assumptions. The following proposition highlights that w^* always exists and is unique for a strictly log-concave function F on $\mathcal{W}_b$.

Proposition 1 (Existence and Uniqueness of Maximum Likelihood)**.** *Suppose the observation graph $\mathcal{G}$ is connected, the choice function $F : \mathbb{R} \to [0,1]$ is strictly log-concave on $[-2b, 2b]$, and Assumption 1 holds. Then, there exists a unique optimal solution $w^* \in \mathcal{W}_b$ satisfying (8).*

The proof is provided in Section IV-B. It follows from Proposition 1 and Gibbs' inequality that when the pairwise comparison model is indeed a $\mathcal{T}_F$ model with weight vector w, then we have $w^* = w$.

C. Minimax Formulation

For any fixed graph $\mathcal{G}$, choice function F, and (universal) constants $\delta, \epsilon > 0$ and $b \geq F^{-1}(1-\delta)/2$, define the sets $\mathcal{M}_0$ and $\mathcal{M}_1(\epsilon)$ of $\mathcal{T}_F$ and pairwise comparison models:

$$\mathcal{M}_0 \triangleq \{P : \text{Assumption 1 holds and } \exists\ w \in \mathcal{W}_b \text{ such that } P = \mathsf{F}(w)\}, \tag{10}$$

$$\mathcal{M}_1(\epsilon) \triangleq \left\{P : \text{Assumption 1 holds and } \inf_{w \in \mathcal{W}_b} \frac{1}{n}\|P - \mathsf{F}(w)\|_{\mathrm{F}} \geq \epsilon\right\}, \tag{11}$$

where $\|\cdot\|_{\mathrm{F}}$ denotes Frobenius norm. Now, we formalize the hypothesis testing problem in (1) as:

$$\begin{aligned} H_0 &: \ \mathcal{Z} \sim P \in \mathcal{M}_0, \\ H_1 &: \ \mathcal{Z} \sim P \in \mathcal{M}_1(\epsilon). \end{aligned} \tag{12}$$

We will discuss the separation distance $\inf_{w\in\mathcal{W}_b}\|P-\mathsf{F}(w)\|_{\mathrm{F}}$ later. For now, note that we only test on the set of observed comparisons $\mathcal{E}$ as it is not possible to determine whether the comparisons on $\mathcal{E}^c$ would conform to a $\mathcal{T}_F$ model or some other pairwise comparison model. Next, for any fixed graph $\mathcal{G}$, choice function F, and constants $\delta, b, \epsilon > 0$, we define the *minimax risk* as:

$$\mathcal{R}(\mathcal{G},\epsilon) \triangleq \inf_{\phi}\Bigg\{\underbrace{\sup_{P\in\mathcal{M}_0}\mathbb{P}_{H_0}(\phi(\mathcal{Z})=1)}_{\mathcal{Z}\sim P\text{ under }H_0}+\underbrace{\sup_{P\in\mathcal{M}_1(\epsilon)}\mathbb{P}_{H_1}(\phi(\mathcal{Z})=0)}_{\mathcal{Z}\sim P\text{ under }H_1}\Bigg\}, \tag{13}$$

where the infimum is taken over all randomized decision rules $\phi(\mathcal{Z})\in\{0,1\}$ (with 0 corresponding to H_0 and 1 to H_1), and $\mathbb{P}_{H_0}$ and $\mathbb{P}_{H_1}$ denote the probability measures under hypotheses H_0 and H_1, respectively. Intuitively, this risk minimizes the sum of the worst-case type I and type II errors. Finally, we define the *critical threshold* of the hypothesis testing problem in (12) as the smallest value of ϵ for which the minimax risk is bounded by $\frac{1}{2}$ (cf. [39]):

$$\varepsilon_{\mathsf{c}} \triangleq \inf\left\{\epsilon>0 : \mathcal{R}(\mathcal{G},\epsilon)\le\frac{1}{2}\right\}. \tag{14}$$

Note that the constant $\frac{1}{2}$ here is arbitrary and can be replaced by any constant in $(0,1)$.

III. Main Results

In this section, we present the main results of the paper. We first show that our notion of separation distance can be simplified for analysis, then proceed to bound the critical threshold and minimax risk, and finally, establish type I and II error bounds in the sequential setting.

A. Separation Distance and Test Statistic

Recall that to formalize (12), we defined the *separation distance* of a pairwise comparison model P to the class of $\mathcal{T}_F$ models as $\inf_{w\in\mathcal{W}_b}\|P-\mathsf{F}(w)\|_{\mathrm{F}}$ (for fixed F). To make this separation distance more amenable to theoretical analysis, we approximate it in the next theorem with the simpler quantity $\|P-\mathsf{F}(w^*)\|_{\mathrm{F}}$, where w^* is given in (8).

Theorem 1 (Separation Distance to $\mathcal{T}_F$ Models)**.** *Let P be a pairwise comparison matrix satisfying Assumption 1. Then, there exists a universal constant $c_1>0$ (that does not depend on n) such that the separation distance between P and the class of $\mathcal{T}_F$ models satisfies*

$$c_1\|P-\mathsf{F}(w^*)\|_{\mathrm{F}} \le \inf_{w\in\mathcal{W}_b}\|P-\mathsf{F}(w)\|_{\mathrm{F}} \le \|P-\mathsf{F}(w^*)\|_{\mathrm{F}},$$

where w^* is given by (8).

The proof is provided in Section IV-C. The upper bound is immediate, and the lower bound utilizes the information-theoretic bounds between f-divergences.

1) Test Statistic: We now introduce our test statistic based on the approximation derived in Theorem 1. First, we partition the observed comparison data $\mathcal{Z}$ into two (roughly) equal parts $\mathcal{Z}_1=\{Z_{ij}^m : (i,j)\in\mathcal{E},\, m\in[\lfloor k_{ij}/2\rfloor]\}$ and $\mathcal{Z}_2=\mathcal{Z}\setminus\mathcal{Z}_1$. The first half of the dataset $\mathcal{Z}_1$ is used to estimate the parameters $\hat{w}$ as shown in (7). Then, we use $\mathcal{Z}_2$ to calculate the *test statistic* T via

$$T \triangleq \sum_{(i,j)\in\mathcal{E}}\left(\frac{Z_{ij}(Z_{ij}-1)}{k'_{ij}(k'_{ij}-1)}+F(\hat{w}_i-\hat{w}_j)^2-2F(\hat{w}_i-\hat{w}_j)\frac{Z_{ij}}{k'_{ij}}\right)\mathbb{1}_{k'_{ij}>1}, \tag{15}$$

where $k'_{ij}=k_{ij}-\lfloor k_{ij}/2\rfloor$, $Z_{ij}=\sum_{m>\lfloor k_{ij}/2\rfloor}Z_{ij}^m$ is computed as before but using only the samples in $\mathcal{Z}_2$, and $\mathbb{1}_{\mathcal{A}}$ denotes the indicator function of $\mathcal{A}$. By construction, if $\hat{w}=w^*$, then $\mathbb{E}[T]=\|P-\mathsf{F}(w^*)\|_{\mathrm{F}}^2$. Hence, T is constructed by plugging in $\hat{w}$ in place of w^* in an unbiased estimator of $\|P-\mathsf{F}(w^*)\|_{\mathrm{F}}^2$. Our proposed *hypothesis test thresholds T to determine the unknown hypothesis* (H_1 is selected if T exceeds a certain threshold). We will discuss analytical expressions for the threshold below and a data-driven manner of determining the threshold in Section VIII.

B. Upper Bound on Critical Threshold

In this section, we make the simplifying assumption that $k_{ij} = 2k$ (with $k \in \mathbb{N}$) for all $(i,j) \in \mathcal{E}$. The ensuing theorem proved in Section V establishes an upper bound on the critical threshold of the hypothesis testing problem defined in (12).

Theorem 2 (Upper Bound on Critical Threshold)**.** *Consider the hypothesis testing problem in (12), and assume that Assumption 1 holds and $k \geq 2$. Then, there exists a constant $c_2 > 0$ such that the critical threshold defined in (14) is upper bounded by*

$$\varepsilon_{\mathsf{c}}^2 \leq \frac{c_2}{nk}.$$

In analysis we select H_1 if $T > \gamma \frac{n}{k}$ and H_0 otherwise, where γ is an appropriate constant independent of n, k (see (37)). The analysis relies on establishing non-trivial error bounds (in $\|\cdot\|_L$ seminorm) for parameter estimation of $\mathcal{T}_F$ models when the data is generated by a general pairwise comparison model, which is not necessarily a GTM (i.e., deriving error bounds under a potential model mismatch). The error bounds allow us to prove bounds on the mean and variance of the test statistic T under both hypotheses H_0 and H_1. Then, using Chebyshev's inequality, we can bound the probabilities of error of our test under each of the hypotheses, which induces an upper bound on the critical threshold.

We also note that in the special case where $\mathcal{T}_F$ is a BTL model, our upper bound on ε_{c} recovers the bound in [42] for complete graphs. But our likelihood-based proof is quite different to the spectral ideas in [42]. Finally, we present the key ℓ^2-error bounds for parameter estimation when data is generated by a general pairwise comparison model needed to prove Theorem 2 (as mentioned above).

Theorem 3 (Error Bounds for Parameter Estimation)**.** *Consider any pairwise comparison model satisfying Assumption 1 with w^* given by (8) and $\hat{w}$ constructed according to (7) from data generated by the model. Then, for some constant $c_3 > 0$, the following tail bound holds on the estimation error of w^*:*

$$\forall t \geq 1, \ \ \mathbb{P}\left(\|\hat{w} - w^*\|_L^2 \geq \frac{c_3 n \beta^2}{\alpha^2 k F(-2b)^2}\, t \right) \leq e^{-t},$$

where α is defined in (9). Moreover, for any $p \geq 1$, there exists a p-dependent constant $c(p) > 0$ such that the expected pth moment of the error is bounded by

$$\mathbb{E}[\|\hat{w} - w^*\|_L^p] \leq \left(\frac{c(p) n \beta^2}{\alpha^2 k F(-2b)^2} \right)^{\frac{p}{2}}.$$

The proof is provided in Section IV-D. In the special case where the pairwise comparison model is a GTM, our bounds recover the bounds derived in [14, Theorem 3] up to constants. However, our result is much more general because it holds for any pairwise comparison model, which requires careful formulation and development of the proof technique.

These error bounds can be easily converted into ℓ^2-error bounds using the relation $\|\hat{w} - w^*\|_L^2 \geq \lambda_2(L) \|\hat{w} - w^*\|_2^2$, where $\lambda_2(L)$ is the second smallest eigenvalue of the Laplacian L. It is worth emphasizing the dependence of the error bound on the topology of the observation graph $\mathcal{G}$. The connectedness of the graph ensures $\lambda_2(L) > 0$, and the value of $\lambda_2(L)$ is well-known for various classes of graphs:

- For a complete graph on n nodes, $d_{\max} = n - 1$ and $\lambda_2(L) = n$, yielding an ℓ^2-error bound of $O(1/k)$.
- For a d-regular spectral expander graph with constant d, $d_{\max} = d$ and $\lambda_2(L) \geq d - 2\sqrt{d}$ [43], yielding an ℓ^2-error bound of $O(1/k)$.
- For a single cycle graph on n nodes, $d_{\max} = 2$ and $\lambda_2(L) = \Theta(1/n^2)$, yielding an ℓ^2-error bound of $O(\frac{n^3}{k})$.
- For a two-dimensional $\sqrt{n} \times \sqrt{n}$ toroidal grid (formed by the Cartesian product of two cycles of length $\sqrt{n}$), $d_{\max} = 4$ and $\lambda_2(L) = \Theta(1/n)$, yielding an ℓ^2-error bound of $O(\frac{n^2}{k})$.

C. Information-Theoretic Lower Bounds

We now establish information-theoretic lower bounds on the minimax risk and critical threshold for the hypothesis testing problem in (12). For simplicity and analytical tractability, assume that $k_{ij} = k \in \mathbb{N}$ for all $(i,j) \in \mathcal{E}$, and assume that the observation graph $\mathcal{G}$ is *super-Eulerian* [44], i.e., it has an Eulerian spanning sub-graph $\tilde{\mathcal{G}} = ([n], \tilde{\mathcal{E}})$ so that every vertex of $\tilde{\mathcal{G}}$ has even degree. Then, $\tilde{\mathcal{G}}$ has a *cycle decomposition* $\mathcal{C}$ by Veblen's theorem [40], [45], where $\mathcal{C}$ is a collection of simple cycles σ that partitions the undirected edges of $\tilde{\mathcal{G}}$. The ensuing theorem, proved in Section VI, presents our minimax risk lower bound.

TABLE I: Bounds in this work on critical threshold ε_{c} for various induced observation graphs, where n represents the number of agents and k is the number of comparisons between agents per pair.

	Complete graph	d-regular graph	Single cycle	Toroidal grid
Upper bound	$O\left(\frac{1}{\sqrt{nk}}\right)$	$O\left(\frac{1}{\sqrt{nk}}\right)$	$O\left(\frac{1}{\sqrt{nk}}\right)$	$O\left(\frac{1}{\sqrt{nk}}\right)$
Lower bound	$\Omega\left(\frac{1}{\sqrt{nk}}\right)$	$\Omega\left(\frac{1}{\sqrt{n^2k}}\right)$	$\Omega\left(\frac{1}{\sqrt{n^2k}}\right)$	$\Omega\left(\frac{1}{\sqrt{n^{7/4}k}}\right)$

Theorem 4 (Minimax Lower Bound)**.** *Consider the hypothesis testing problem in (12) and assume that the observation graph $\mathcal{G}$ is super-Eulerian with spanning Eulerian sub-graph $\tilde{\mathcal{G}}$. Then, there exists a constant $c_4 > 0$ such that for any $\epsilon > 0$, the minimax risk in (13) is lower bounded by*

$$\mathcal{R}(\mathcal{G},\epsilon) \geq 1 - \frac{1}{2}\sqrt{\exp\left(\frac{c_4 k^2 n^4 \epsilon^4}{|\tilde{\mathcal{E}}|^2}\sum_{\sigma\in\mathcal{C}}|\sigma|^2\right) - 1}\,,$$

where $|\sigma|$ denotes the length of a cycle $\sigma \in \mathcal{C}$, and $\mathcal{C}$ is the cycle decomposition of $\tilde{\mathcal{G}}$.

Our approach utilizes the Ingster-Suslina method [46], which is similar to *Le Cam's method*, but provides a lower bound by considering a cleverly chosen point and a mixture on the parameter space instead of just two points. Our specific construction is inspired by the technique introduced in [40], which establishes a lower bound for testing of IIA assumption for the BTL model and for Eulerian graph structures. We extend their approach in three significant ways. First, we generalize their method to accommodate any GTM rather than just the BTL model. Second, we use a different technique based on Theorem 1 to lower bound separation distance from the class of $\mathcal{T}_F$ models. Moreover, our work quantifies separation using Frobenius norm instead of sums of total variation distances. Third, our argument holds for a broader class of graphs, namely, super-Eulerian graphs. Note that the question of algorithmically constructing Eulerian sub-graphs of graphs has been widely studied [47].

The following proposition simplifies Theorem 4 to obtain lower bounds on the critical threshold for several classes of graphs.

Proposition 2 (Lower Bounds on Critical Threshold)**.** *Under the assumptions of Theorem 4, the following lower bounds hold for the critical threshold defined in (14):*
1) If $\mathcal{G}$ is a complete graph with odd n vertices, then $\varepsilon_{\mathsf{c}}^2 = \Omega(1/nk)$.
2) If $\mathcal{G}$ is a d-regular graph with constant $d \geq 2$, then $\varepsilon_{\mathsf{c}}^2 = \Omega(1/n^2k)$.
3) If $\mathcal{G}$ is a single cycle graph with n vertices, then $\varepsilon_{\mathsf{c}}^2 = \Omega(1/n^2k)$.
4) If $\mathcal{G}$ is a two-dimensional $\sqrt{n} \times \sqrt{n}$ toroidal grid on n vertices formed by the Cartesian product of two cycles of length $\sqrt{n}$, then $\varepsilon_{\mathsf{c}}^2 = \Omega(1/n^{7/4}k)$.

The proof of Proposition 2 involves calculating the number of simple cycles and the individual cycle lengths in the cycle decompositions $\mathcal{C}$ and is provided in Section VI-B. The lower bounds on ε_{c} are then obtained from Theorem 4. We remark that our minimax upper and lower bounds on ε_{c} match for the complete graph case, demonstrating the minimax optimality of the threshold's scaling (up to constant factors). Moreover, they also match with respect to k for other classes of graphs as well. It is worth mentioning that in the special case of BTL models with single cycle graphs, our lower bound on ε_{c} improves the high-level scaling behavior in [40] from $\Omega(1/\sqrt{n^3k})$ to $\Omega(1/\sqrt{n^2k})$ (when ε_{c} is quantified in terms of Frobenius norm). Lastly, we remark that for single cycle graphs, the gap between the upper and lower bounds in terms of n intuitively holds because our lower bounds become larger when there are more cycles in $\mathcal{C}$, which is only 1 in this case.

D. Upper Bounds on Type I and II Error Probabilities

To complement the minimax risk lower bound in Theorem 4, we establish upper bounds on the extremal type I and II error probabilities. We will do this in the *sequential* setting, where data is observed incrementally—a common practical scenario which subsumes the standard fixed sample-size setting, cf. [48], [49]. In the sequential testing framework, at each time step, we observe a single "i vs. j" comparison for every $(i,j) \in \mathcal{E}$. (The subsequent analysis can be extended to a general setting where we observe only one comparison for some pair $(i,j) \in \mathcal{E}$ or even a variable number of comparisons at every time step.) At time $k_1 + k$ with $k_1, k \in \mathbb{N}$, we define $T^{k_1,k}$ to be the value of the test statistic T in (15), where comparisons from k_1 time-steps have been used to build the dataset $\mathcal{Z}_1$ to estimate parameters using

$\hat{w}$, and k time-steps have been used to build the dataset $\mathcal{Z}_2$ to calculate the statistic T. Note that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ no longer need to be similar in size. Then, we can decide based on thresholding $T^{k_1,k}$ (see Theorem 5) whether to collect more data or stop and reject H_0 while controlling the probabilities of error. If the testing process ends without rejecting H_0, then we can accept H_0. A key observation underlying our analysis is the following *reverse martingale* property (see, e.g., [48], [49]).

Proposition 3 (Reverse Martingale)**.** *Fix any $k_1 \in \mathbb{N}$, and let $\mathcal{F}_k = \bigotimes_{(i,j)\in\mathcal{E}} \sigma(\sum_{m=k_1+1}^{k_1+k} Z_{ij}^m, Z_{ij}^{k_1+k+1}, Z_{ij}^{k_1+k+2}, \dots)$ be a non-increasing sequence of σ-algebras, where $\bigotimes$ denotes the product σ-algebra. Then, the sequence of test-statistics $\{T^{k_1,k} : k \geq 2\}$ is a reverse martingale with respect to reverse filtration $\{\mathcal{F}_k : k \geq 2\}$, i.e., for $k \geq 2$, $T^{k_1,k}$ is $\mathcal{F}_k$-measurable and $\mathbb{E}\big[T^{k_1,k}|\mathcal{F}_{k+1}\big] = T^{k_1,k+1}$.*

The proof is presented in Section VII-A. This observation allows us to develop *time-uniform bounds* in terms of k on type I and type II error probabilities, i.e., they hold for all k larger than a constant. The next theorem, proved in Section VII-C, presents our type I and type II error probability bounds.

Theorem 5 (Type I and Type II Error Probability Bounds)**.** *Under the sequential setting discussed above, the following bounds hold on the extremal type I and type II error probabilities. There exist constants c_5, c_6, c_7, c_8 such that for all $t \geq 1$, $\nu \in (0, 1/e)$, $k_1 \in \mathbb{N}$ and $\epsilon \geq c_5 t^{\frac{1}{2}}/\sqrt{nk_1}$, we have*

$$\sup_{P\in\mathcal{M}_0} \mathbb{P}_{H_0}\left(\exists k \geq 2, T^{k_1,k} \geq c_6 t\frac{n}{k_1} + \frac{c_7|\mathcal{E}|^{\frac{1}{2}}\ell_{k,\nu}}{k} + c_8\sqrt{\frac{nt\ell_{k,\nu}}{k_1 k}}\right) \leq \nu + e^{-t},$$

$$\sup_{P\in\mathcal{M}_1(\epsilon)} \mathbb{P}_{H_1}\left(\exists k \geq 2,\, T^{k_1,k} - (D - c_5 t^{\frac{1}{2}} n^{\frac{1}{2}}/k_1^{\frac{1}{2}})^2 \leq -\frac{c_7|\mathcal{E}|^{\frac{1}{2}}\ell_{k,\nu}}{k} - (4D + c_8 t^{\frac{1}{2}} n^{\frac{1}{2}}/k_1^{\frac{1}{2}})\sqrt{\frac{\ell_{k,\nu}}{k}}\right) \leq \nu + e^{-t},$$

where $D \triangleq \|P - \mathsf{F}(w^*)\|_{\mathrm{F}}$ and $\tilde{\varepsilon} \triangleq \sqrt{n/k_1}$ and $\ell_{k,\nu} \triangleq \log\big(3.5\log^2(k)/\nu\big)$.

We now make several remarks. Firstly, our error probability bounds encode the scalings of the thresholds to accept or reject H_0 (see (37)). Secondly, our bounds hold regardless of how the decision-maker assigns data collected at different time-steps to $\mathcal{Z}_1$ and $\mathcal{Z}_2$. Moreover, they provide insights on how to split the data based on the topology of the observation graph, e.g., the bounds suggest an equal split of the data for complete graphs, whereas for a single cycle, achieving better Type I error control favors a larger value of k_1. To illustrate this and help parse Theorem 5, we present corollaries of Theorem 5 for the complete and single cycle graph cases in Appendix A.

Thirdly, our bounds clearly hold in the non-sequential fixed sample-size setting, as we can just fix a particular value of k. Hence, adding the two extremal probabilities of error yields upper bounds on the minimax risk. Notably, the proof of Theorem 5 requires us to develop a time-uniform version of the well-known Hanson-Wright inequality [50] specialized for our setting (see Lemma 5 in Section VII-B). Additionally, as an intermediate step in the proof, we also obtain time-uniform *confidence intervals* under the null hypothesis H_0, as demonstrated in the following proposition.

Proposition 4 (Confidence Interval for $T^{k_1,k}$)**.** *Suppose $\hat{w}$ is estimated as in (7) from the comparisons over k_1 time-steps. Then, there exists a constant $c_6 > 0$ such that for all $\nu \in (0, 1/e)$ and $k_1 \in \mathbb{N}$,*

$$\mathbb{P}_{H_0}\left(\exists k \geq 2, T^{k_1,k} \geq \|\mathsf{F}(\hat{w}) - \mathsf{F}(w^*)\|_{\mathrm{F}}^2 + c_7\frac{\sqrt{|\mathcal{E}|\ell_{k,\nu}}}{k} + 4\|\mathsf{F}(\hat{w}) - \mathsf{F}(w^*)\|_{\mathrm{F}}\sqrt{\frac{\ell_{k,\nu}}{k}}\right) \leq \nu.$$

Proposition 4 is established in Section VII-B. We remark that the distribution of $\|\mathsf{F}(\hat{w}) - \mathsf{F}(w^*)\|_{\mathrm{F}}$ above can be approximated either by leveraging the asymptotic normality of $\hat{w} - w^*$ [35], [36], or by utilizing bootstrapping techniques; this gives $(1-2\nu)$ time-uniform confidence intervals. Additionally, the constant c_7 here is also the constant in our specialized version of Hanson-Wright inequality (noted above) and, for our setting, can be approximated via simulations. An empirical investigation into estimating the constant c_6 and the subsequent confidence intervals can be found in Appendix B.

IV. Proofs of Proposition 1 and Theorems 1 and 3

A. Additional Notation and Preliminaries

We begin by introducing some additional notations and discuss some necessary preliminaries that will used throughout our proofs. To simplify our notation, we use $\hat{l}(w)$ to denote $l(w; \{\hat{p}_{ij} : (i,j) \in \mathcal{E}\})$ where $\hat{p}_{ij}$ are computed based on the partitioned dataset $\mathcal{Z}_1$. Similarly, we use $l^*(w)$ to denote $l(w; \{p_{ij} : (i,j) \in \mathcal{E}\})$ where p_{ij} are the actual underlying pairwise comparison probabilities. When $w = w^*$ or $w = \hat{w}$, we simplify the notation by using F and $\hat{\mathsf{F}}$ to denote the matrices $\mathsf{F}(w^*)$ and $\mathsf{F}(\hat{w})$ (cf. (4)) respectively, for brevity. We say that a random variable X is μ^2-sub-Gaussian, if it satisfies the condition, $\log(\mathbb{E}[\exp(sX)]) \leq \mu^2 s^2/2$ for all $s \in \mathbb{R}$. Notably, $\hat{w}$ is computed as in (7) even though the data may not conform to an underlying $\mathcal{T}_F$ model. Throughout the appendices, we denote various constants using overlapping labels, such as $c, c_1, c_2, \ldots$ to simplify our notation and facilitate readability. Moreover, we define $\mathcal{E}_+$ to denote the set $\{(i,j) \in \mathcal{E} : j > i\}$.

1) Prelimiaries: Recall that, as defined in (8), w^* represents the weights of a $\mathcal{T}_F$ model that best approximates the pairwise comparison model $\{p_{ij}, (i,j) \in \mathcal{E}\}$. Now, we show that any pairwise comparison model can be converted into its skew-symmetric counterpart $\left\{\frac{p_{ij}+1-p_{ji}}{2}, (i,j) \in \mathcal{E}\right\}$ such that both of them share the same optimal weights.

Lemma 1 (Skew-Symmetrized Model)**.** *For a symmetric edge set $\mathcal{E}$, the pairwise model $\{p_{ij} : (i,j) \in \mathcal{E}\}$ and and its skew-symmetric counterpart $\left\{\frac{p_{ij}+1-p_{ji}}{2} : (i,j) \in \mathcal{E}\right\}$ has the have the same optimal $\mathcal{T}_F$ weights w^* as defined in (8).*

Proof. Note that the weighted negative log-likelihood objective can be written as

$$\begin{aligned} l^*(w) &= \underset{w \in \mathcal{W}: w^{\mathrm{T}}\mathbf{1}=0}{\arg\min} \; -\sum_{(i,j)\in\mathcal{E}} p_{ij} \log(F(w_i - w_j)) + (1-p_{ij}) \log(1 - F(w_i - w_j)) \\ &\overset{\zeta_1}{=} \underset{w \in \mathcal{W}: w^{\mathrm{T}}\mathbf{1}=0}{\arg\min} \; -\sum_{(i,j)\in\mathcal{E}} p_{ij} \log(1 - F(w_j - w_i)) + (1-p_{ij}) \log(F(w_j - w_i)) \\ &\overset{\zeta_2}{=} \underset{w \in \mathcal{W}: w^{\mathrm{T}}\mathbf{1}=0}{\arg\min} \; -\sum_{(i,j)\in\mathcal{E}} \left(\frac{p_{ij}+1-p_{ji}}{2}\right) \log(F(w_i - w_j)) + \left(\frac{1-p_{ij}+p_{ji}}{2}\right) \log(F(w_j - w_i)) \end{aligned}$$

where ζ_1 follows since $F(-x) = 1 - F(x)$ and ζ_2 follows by adding the first two equations and dividing by two. □

Therefore, for any pairwise comparison model $\{p_{ij} : (i,j) \in \mathcal{E}\}$, we can define its skew-symmetrized counterpart $\{q_{ij} : (i,j) \in \mathcal{E}\}$, where:

$$\forall\ (i,j) \in \mathcal{E}, q_{ij} \triangleq \frac{p_{ij}+1-p_{ji}}{2}. \tag{16}$$

We call these transformed probabilities q_{ij} as skew-symmetrized probabilities because we have $q_{ij} + q_{ji} = 1$, and thereby this transformation effectively removes any distinctions between "i vs. j" and "j vs. i" comparisons. Also, note that for any pairwise comparison model satisfying Assumption 1 its skew-symmetrized model also satisfies it. In a similar manner, we can define $\hat{q}_{ij} = \frac{\hat{p}_{ij}+1-\hat{p}_{ji}}{2}$ as the skew-symmetrized version of the empirical probabilities. With this notation in place, we are ready to state the proof of Proposition 1 below.

B. Proof of Proposition 1

Uniqueness. The uniqueness of w follows directly from the strong log-concavity of $F(\cdot)$. This is because if $v^*, w^* \in \mathcal{W}_b$ are any two non-unique solutions of (8) such that $l^*(v^*) = l^*(w^*)$, then by strong log concavity of F and the fact that $q_{ij} > 0$ for $(i,j) \in \mathcal{E}$ along with connectedness of graph, for any $\theta \in (0,1)$, we have

$$\begin{aligned} \theta l^*(v^*) + (1-\theta) l^*(w^*) &= -2\theta \sum_{(i,j)\in\mathcal{E}} q_{ij} \log\big(F\big(v_i^* - v_j^*\big)\big) - 2(1-\theta) \sum_{(i,j)\in\mathcal{E}} q_{ij} \log\big(F\big(w_i^* - w_j^*\big)\big) \\ &> -2 \sum_{(i,j)\in\mathcal{E}} q_{ij} \log\left(F\big(\theta\big(v_i^* - v_j^*\big) + (1-\theta)\big(w_i^* - w_j^*\big)\big)\right) \\ &= \quad l^*(\theta v^* + (1-\theta) w^*). \end{aligned}$$

This gives a contradiction since $\theta v^* + (1-\theta) w^*$ achieves a higher likelihood (or a lower objective value). The existence of w^* under Assumption 1 and finite b follows since the optimization of convex function is being performed over a compact set. Below we show a proof of existence even when the parameter $b = \infty$.

Existence. Now, we will utilize the connectedness of graph $\mathcal{G}$ and Assumption 1 to show the existence of w^*. Define a sequence $\{w^{(m)} \in \mathbb{R}^n : m \in \mathbb{N} \cup \{0\}\}$ as the

$$w^{(m)} = \operatorname{argmin}_{\substack{w_1=0: \\ \|w\|_\infty \leq m}} l^*(w).$$

Clearly $w^{(m)}$ exists as the optimization of a convex function $l^*(\cdot)$ is being performed over a compact set. Define the following sets as the components of w that potentially diverge to ∞:

$$S_+ = \left\{ i \in [n] : \limsup_m \ \big(w^{(m)}\big)_i = +\infty \right\}, \quad S_- = \left\{ i \in [n] : \liminf_m \ \big(w^{(m)}\big)_i = -\infty \right\}.$$

We will show that $S_+ = S_- = \varnothing$. Notably, if $S_+ \neq \varnothing$, then we consider the partition of $[n]$ as $S_+ \cup S_+^c$. Clearly, $1 \in S_+^c \neq \varnothing$. Since the observation graph $\mathcal{G}$ is connected, for some $i \in S_+$ there exists $j \in S_+^c$ such that $q_{ji} > 0$ (by Assumption 1). This implies that $-q_{ji} \log\Big(F(w_j^{(m)} - w_i^{(m)})\Big) \to +\infty$ as $m \to +\infty$. Hence, we can find a constant $A > 0$ such that on the set $\{w_i - w_j \geq A\}$, we have

$$-q_{ji} \log(F(w_i - w_j)) > l^*(w^{(0)}).$$

Equivalently, for any w with $w_i - w_j > A$, we have

$$l^*(w) \geq -q_{ji} \log(F(A)) > l^*(w^{(0)}) \geq l^*(w^{(m)}),$$

where the first inequality follows since each term in $l^*(\cdot)$ is non-negative. Therefore, we must have $w_i^{(m)} \leq w_j^{(m)} + A$ for all $k \in \mathbb{N}$. Since $i \in S_+$, it follows that $j \in S_+$ by definition, which contradicts the assumption that $j \in S_+^c$. Hence, we conclude that $S_+ = \emptyset$. A similar argument shows that $S_- = \varnothing$. The fact that $S_+ = S_- = \varnothing$ implies that the sequence $\{w^{(m)} : m \in \mathbb{N} \cup \{0\}\}$ admits a convergent subsequence, which proves the existence of w^*. □

Limitations of our assumptions. Both GTM and pairwise comparison models assume that different comparisons between the agents are independent and the comparison probabilities remain constant over time. However, this is rarely the case in real-world settings, where comparisons are often correlated and the underlying probabilities evolve with time. Secondly, our testing framework assumes the existence of parameters δ in Assumption 1 about the true comparison probabilities. However, such constants are typically unknown in practice and selected based on intuition. Third, it is not clear how large should the parameter b be as compared to $F^{-1}(1-\delta)/2$ so that it allows us to estimate the weights from empirical probabilities $\hat{p}_{ij}$, without introducing 'biases' towards smaller values and at the same time being large enough to give a reasonable approximation of separation distance with $b = \infty$.

C. *Proof of Theorem 1*

The upper bound is trivial to prove

$$\inf_{w \in \mathcal{W}_b} \|P - \mathsf{F}(w)\|_{\mathrm{F}} \leq \|P - \mathsf{F}(w^*)\|_{\mathrm{F}} = \|P - \mathsf{F}\|_{\mathrm{F}},$$

where F is the pairwise probability matrix associated with the optimal weights. Now to prove the lower bound, observe that

$$\begin{aligned}
&\inf_{w \in \mathcal{W}_b} \sum_{(i,j)\in\mathcal{E}} (p_{ij} - F(w_i - w_j))^2 \\
&\quad\geq \inf_{w \in \mathcal{W}_b} F(-2b)(1-F(-2b)) \sum_{(i,j)\in\mathcal{E}} \frac{(p_{ij} - F(w_i - w_j))^2}{F(w_i - w_j)(1 - F(w_i - w_j))} \\
&\quad\overset{\zeta_1}{=} c \inf_{w \in \mathcal{W}_b} \sum_{(i,j)\in\mathcal{E}} \chi^2(\mathsf{Bernoulli}(p_{ij}) \| \mathsf{Bernoulli}(F(w_i - w_j))) \\
&\quad\overset{\zeta_2}{\geq} c \inf_{w \in \mathcal{W}_b} \sum_{(i,j)\in\mathcal{E}} D_{\mathsf{KL}}(\mathsf{Bernoulli}(p_{ij}) \| \mathsf{Bernoulli}(F(w_i - w_j))) \\
&\quad= c \inf_{w \in \mathcal{W}_b} \sum_{(i,j)\in\mathcal{E}} p_{ij} \log\left(\frac{p_{ij}}{F(w_i - w_j)}\right) + (1-p_{ij}) \log\left(\frac{1-p_{ij}}{1 - F(w_i - w_j)}\right) \\
&\quad\overset{\zeta_3}{=} c \sum_{(i,j)\in\mathcal{E}} p_{ij} \log\left(\frac{p_{ij}}{F(w_i^* - w_j^*)}\right) + (1-p_{ij}) \log\left(\frac{1-p_{ij}}{1 - F(w_i^* - w_j^*)}\right) \\
&\quad\overset{\zeta_4}{\geq} 2c \sum_{(i,j)\in\mathcal{E}} \|\mathsf{Bernoulli}(p_{ij}) - \mathsf{Bernoulli}(F(w_i^* - w_j^*))\|_{\mathsf{TV}}^2 \\
&\quad= 2c \sum_{(i,j)\in\mathcal{E}} (p_{ij} - F(w_i^* - w_j^*))^2,
\end{aligned}$$

where, in ζ_1 we set $c = F(-2b)(1 - F(-2b))$ and $\chi^2(\cdot||\cdot)$ denotes the χ^2-divergence between two Bernoulli random variables and in ζ_2 we utilize the fact that $\chi^2(R||Q) \geq D_{\mathsf{KL}}(R||Q)$ for two distributions R and Q and where $D_{\mathsf{KL}}(\cdot||\cdot)$ denotes the Kullback-Leibler (KL) divergence between two distributions, ζ_3 follows since w^* are the optimal weights maximizing (8) for the $\mathcal{T}_F$ model. Finally, ζ_4 follows by Pinsker's inequality, $D_{\mathsf{KL}}(R||P) \geq 2\|R - P\|_{\mathsf{TV}}^2$, and thereby completes the proof. □

D. Proof of Theorem 3

We begin by recalling the definition of $\hat{w} \in \arg\min_{w \in \mathcal{W}_b} \hat{l}(w)$ in terms of the symmetrized probabilities q_{ij} defined in (16) as

$$\hat{l}(w) = -2 \sum_{(i,j)\in\mathcal{E}_+} \hat{q}_{ij} \log F(w_i - w_j) + (1-\hat{q}_{ij}) \log(1 - F(w_i - w_j)).$$

Observe that since $\hat{w}$ is an optimal solution and w^* is a feasible point for the problem in (7), therefore we have $\hat{l}(\hat{w}) \leq \hat{l}(w^*)$. Moreover, since w^* is the optimal solution of a convex function $l^*(w)$, therefore we have the optimality condition $\nabla l^*(w^*)^{\mathrm{T}}(w - w^*) \geq 0$ for all $w \in \mathcal{W}_b$. Now, by subtracting the quantity $\nabla \hat{l}(w^*)^{\mathrm{T}}(\hat{w} - w^*)$ from both sides of $\hat{l}(\hat{w}) \leq \hat{l}(w^*)$ gives

$$\begin{aligned} \hat{l}(\hat{w}) - \hat{l}(w^*) - \nabla \hat{l}(w^*)^{\mathrm{T}}(\hat{w} - w^*) &\leq -\nabla \hat{l}(w^*)^{\mathrm{T}}(\hat{w} - w^*) && (17)\\ &\overset{\zeta_1}{\leq} -(\nabla \hat{l}(w^*) - \nabla l^*(w^*))^{\mathrm{T}}(\hat{w} - w^*) \\ &\overset{\zeta_2}{\leq} \|\nabla \hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger} \|\hat{w} - w^*\|_L, && (18)\end{aligned}$$

where ζ_1 follows since $\nabla l^*(w^*)^{\mathrm{T}}(w - w^*) \geq 0$ for all $w \in \mathcal{W}$ and ζ_2 follows from [14, Lemma 16] where $\|\cdot\|_L$ is the semi-norm induced by the Laplacian matrix L of graph $\mathcal{G}$ and $L^\dagger$ is the Moore-Penrose pseudoinverse of L. Now observe that by chain rule, the Hessian of $\hat{l}$ is given by

$$\nabla^2 \hat{l}(w) = -2 \sum_{(i,j)\in\mathcal{E}_+} \left(\hat{q}_{ij} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(F(t))|_{t = w^{\mathrm{T}} x_{ij}} + (1 - \hat{q}_{ij}) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(1 - F(t))|_{t = w^{\mathrm{T}} x_{ij}} \right) x_{ij} x_{ij}^{\mathrm{T}},$$

Since by our assumption that $F(t)$ is α-strongly log-concave on the set $[-2b, 2b]$, this implies $-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(F(t)) \geq \alpha$. Moreover, since $F(-t) = 1 - F(t)$, we also have $-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log(1 - F(t)) \geq \alpha$ for all $t \in [-2b, 2b]$. Therefore, for any $v \in \mathbb{R}^n$ with $v^{\mathrm{T}}\mathbf{1} = 0$, we have

$$v^{\mathrm{T}} \nabla^2 \hat{\ell}(w^*) v \geq 2\alpha \|Xv\|_2^2 = 2\alpha \|v\|_L^2.$$

Thus, by definition of strong-convexity, the left side of (17) can be lower bounded by $\alpha\|\hat{w} - w^*\|_L^2$. Therefore, utilizing the bound in (18), we obtain the following inequality

$$\alpha \|\hat{w} - w^*\|_L^2 \leq \|\nabla \hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger} \|\hat{w} - w^*\|_L.$$

Cancelling $\|\hat{w} - w^*\|_L$ leads to the following error bound on $\|\hat{w} - w^*\|_L$ as

$$\|\hat{w} - w^*\|_L \leq \frac{1}{\alpha} \|\nabla \hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}. \tag{19}$$

Since, both $\hat{w}, w^*$ satisfy $\hat{w}^{\mathrm{T}}\mathbf{1} = 0$ and ${w^*}^{\mathrm{T}}\mathbf{1} = 0$, this yields the following upper bound on $\|\hat{w} - w^*\|_L$

$$\|\hat{w} - w^*\|_L^2 \leq \frac{1}{\alpha^2} \|\nabla \hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}^2. \tag{20}$$

Now, it remains to bound the term $\|\nabla \hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}$. Note that we can express the respective quantities as:

$$\begin{aligned} \nabla \hat{l}(w^*) &= -2 \sum_{(i,j)\in\mathcal{E}_+} \left(\hat{q}_{ij} \frac{F'({w^*}^{\mathrm{T}} x_{ij})}{F({w^*}^{\mathrm{T}} x_{ij})} - (1 - \hat{q}_{ij}) \frac{F'({w^*}^{\mathrm{T}} x_{ij})}{1 - F({w^*}^{\mathrm{T}} x_{ij})} \right) x_{ij} \\ &= -2 \sum_{(i,j)\in\mathcal{E}_+} \frac{(\hat{q}_{ij} - F(w_i^* - w_j^*)) F'(w_i^* - w_j^*)}{F(w_i^* - w_j^*)(1 - F(w_i^* - w_j^*))} x_{ij}, && (21)\\ \nabla l^*(w^*) &= -2 \sum_{(i,j)\in\mathcal{E}_+} \frac{(q_{ij} - F(w_i^* - w_j^*)) F'(w_i^* - w_j^*)}{F(w_i^* - w_j^*)(1 - F(w_i^* - w_j^*))} x_{ij}. && (22)\end{aligned}$$

Therefore, subtracting the two equations gives

$$\nabla \hat{l}(w^*) - \nabla l^*(w^*) = -2 \sum_{(i,j)\in\mathcal{E}_+} \frac{(\hat{q}_{ij} - q_{ij}) F'(w_i^* - w_j^*)}{F(w_i^* - w_j^*)(1 - F(w_i^* - w_j^*))} x_{ij} = -2X^{\mathrm{T}} v, \tag{23}$$

where $v \in \mathbb{R}^{|\mathcal{E}|/2}$ is a vector formed by entries v_{ij} for $(i,j) \in \mathcal{E}_+$. and quantities v_{ij} are defined as

$$v_{ij} \triangleq (\hat{q}_{ij} - q_{ij}) \times \frac{F'(w_i^* - w_j^*)}{F(w_i^* - w_j^*)(1 - F(w_i^* - w_j^*))}.$$

Note that the entries of the vector v are independent and have a mean of zero. Furthermore, we also have:

$$\sup_{x \in [-2b, 2b]} \frac{F'(x)}{F(x)(1-F(x))} \le \frac{\beta}{F(-2b)(1-F(-2b))} \triangleq \tilde{\beta}.$$

Additionally, for any $(i,j) \in \mathcal{E}_+$, an application of the Hoeffding's inequality on $\hat{q}_{ij} - q_{ij}$ yields the following tail bound

$$\begin{aligned}\forall\ t > 0,\ \mathbb{P}(|\hat{q}_{ij} - q_{ij}| > t) &= \mathbb{P}\left(\frac{1}{2k}\left|\sum_{m=1}^{k}(Z_{ij}^m - p_{ij}) + \sum_{m=1}^{k}(Z_{ji}^m - p_{ji})\right| > t\right) \\ &\le 2\exp\big(-2kt^2\big).\end{aligned}$$

Consequently, v is a vector whose each entry is independent with zero mean and $\frac{\tilde{\beta}^2}{4k}$-sub-gaussian. Now, observe that we can express $\|\nabla\hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}^2$ in quadratic form as

$$\|\nabla\hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}^2 = 4v^{\mathrm{T}} X L^\dagger X^{\mathrm{T}} v. \tag{24}$$

Now, combining (20) and (24) we can upper-bound $\mathbb{E}[\|\hat{w} - w^*\|_L^2]$ as

$$\begin{aligned}\mathbb{E}[\|\hat{w} - w^*\|_L^2] &\le \frac{1}{\alpha^2}\mathbb{E}[\|\nabla\hat{l}(w^*) - \nabla l^*(w^*)\|_{L^\dagger}^2] \\ &= \frac{4}{\alpha^2}\mathbb{E}[v^{\mathrm{T}} X L^\dagger X^{\mathrm{T}} v] \\ &\le \frac{\tilde{\beta}^2}{k\alpha^2}\operatorname{tr}\big(XL^\dagger X^{\mathrm{T}}\big) = \frac{(n-1)\tilde{\beta}^2}{k\alpha^2},\end{aligned} \tag{25}$$

where tr denotes the trace operator and we have $\operatorname{tr}\big(XL^\dagger X^{\mathrm{T}}\big) = \operatorname{tr}\big(L^\dagger X^{\mathrm{T}} X\big) = \operatorname{tr}\big(L^\dagger L\big) = n-1$. Hence, by an application of Hanson-Wright inequality [50] combined with usage of (20) and (24) as above, we have the following concentration bounds on $\|\hat{w} - w^*\|_L^2$ as

$$\begin{aligned}\forall t > 0, \mathbb{P}\left(\|\hat{w} - w^*\|_L^2 - \frac{n\tilde{\beta}^2}{k\alpha^2} > t\right) &\le 2\exp\left(-c\min\left\{\frac{t^2k^2\alpha^4}{\tilde{\beta}^4\|XL^\dagger X^{\mathrm{T}}\|_{\mathrm{F}}^2}, \frac{tk\alpha^2}{\tilde{\beta}^2\|XL^\dagger X^{\mathrm{T}}\|_2}\right\}\right) \\ &= 2\exp\left(-c\min\left\{\frac{t^2k^2\alpha^4}{\tilde{\beta}^4(n-1)}, \frac{tk\alpha^2}{\tilde{\beta}^2}\right\}\right).\end{aligned}$$

Hence, by a simple calculation, we can conclude that for some universal constant c, we have

$$\text{for all } t \ge 1,\ \mathbb{P}\left(\|\hat{w} - w^*\|_L^2 > t\frac{cn\tilde{\beta}^2}{k\alpha^2}\right) \le e^{-t}. \tag{26}$$

Bounding the pth moment. Let A denote the quantity: $A = \sqrt{cn\tilde{\beta}^2/k\alpha^2}$. Now the bound on the pth moment is obtained by integration and utilizing the tail bound in (26) as

$$\begin{aligned}\mathbb{E}[\|\hat{w} - w^*\|_L^p] &= p\int_0^\infty t^{p-1}\mathbb{P}(\|\hat{w} - w^*\|_L > t)\,\mathrm{d}t \\ &= p\int_0^A t^{p-1}\mathbb{P}(\|\hat{w} - w^*\|_L > t)\,\mathrm{d}t + p\int_A^\infty t^{p-1}\mathbb{P}(\|\hat{w} - w^*\|_L > t)\,\mathrm{d}t \\ &\le \int_0^A t^{p-1}\,\mathrm{d}t + p\int_1^\infty (At)^{p-1}\mathbb{P}(\|\hat{w} - w^*\|_L > tA)A\,\mathrm{d}t \\ &\le A^p + pA^p\int_0^\infty t^{p-1}e^{-\sqrt{t}} \le c(p)A^p.\end{aligned}$$

Substituting the value of A in the above expression completes the proof. □

V. Proof of Theorem 2

We begin by recalling the test statistic T from (15) as

$$T = \sum_{(i,j)\in\mathcal{E}} \frac{Z_{ij}(Z_{ij}-1)}{k'_{ij}(k'_{ij}-1)} + F(\hat{w}_i - \hat{w}_j)^2 - \frac{2Z_{ij}}{k'_{ij}} F(\hat{w}_i - \hat{w}_j).$$

where $\hat{w}$ is calculated based on the data in $\mathcal{Z}_1$ and Z_{ij} are calculated based on the data in $\mathcal{Z}_2$. The expected value of T conditioned on the weights $\hat{w}$ or equivalently conditioned on the data $\mathcal{Z}_1$ is given by

$$\begin{aligned}\mathbb{E}[T \mid \mathcal{Z}_1] &= \sum_{(i,j)\in\mathcal{E}} \mathbb{E}\left[\frac{Z_{ij}(Z_{ij}-1)}{k'_{ij}(k'_{ij}-1)} | \mathcal{Z}_1\right] + F(\hat{w}_i - \hat{w}_j)^2 - 2\mathbb{E}\left[\frac{Z_{ij}}{k'_{ij}} | \ \mathcal{Z}_1\right] F(\hat{w}_i - \hat{w}_j) \\ &\overset{\zeta}{=} \sum_{(i,j)\in\mathcal{E}} p_{ij}^2 + F(\hat{w}_i - \hat{w}_j)^2 - 2p_{ij}F(\hat{w}_i - \hat{w}_j) \\ &= \sum_{(i,j)\in\mathcal{E}} (p_{ij} - F(\hat{w}_i - \hat{w}_j))^2, \end{aligned} \tag{27}$$

where in ζ we have utilized the fact that $\mathbb{E}\left[\frac{Z_{ij}(Z_{ij}-1)}{k'_{ij}(k'_{ij}-1)} | \mathcal{Z}_1\right] = p_{ij}^2$. Hence, the expected value of T is given by

$$\begin{aligned}\mathbb{E}[T] = \mathbb{E}[\mathbb{E}[T|\mathcal{Z}_1]] &= \sum_{(i,j)\in\mathcal{E}} \mathbb{E}[(p_{ij} - F(\hat{w}_i - \hat{w}_j))^2] \\ &= \sum_{(i,j)\in\mathcal{E}} (p_{ij} - F(w_i^* - w_j^*))^2 + \mathbb{E}[(F(w_i^* - w_j^*) - F(\hat{w}_i - \hat{w}_j))^2] \\ &\qquad + 2\mathbb{E}[(p_{ij} - F(w_i^* - w_j^*))(F(w_i^* - w_j^*) - F(\hat{w}_i - \hat{w}_j))] \\ &\le \|P - \mathsf{F}\|_{\mathrm{F}}^2 + \mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2] + 2\|P - \mathsf{F}\|_{\mathrm{F}} \mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}], \end{aligned} \tag{28}$$

where $\mathsf{F}, \hat{\mathsf{F}} \in \mathbb{R}^{n\times n}$ are matrices defined in Section IV-A. In order to find bounds on estimation error (such as terms like $\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^p]$), we will utilize our simplifying assumption that $k_{ij} = 2k$ for all $(i,j) \in \mathcal{E}$. Now the ensuing lemma provides the bounds on the pth moments $\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^p]$.

Lemma 2 (pth Moment Bound). *For matrices F and $\hat{\mathsf{F}}$ defined as in Section IV-A, there exists constant c_p possibly dependent on p, α, β, b such that the following bound hold on the pth moment of Frobenius norm $\mathbb{E}\Big[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^p\Big]$, for $p \ge 1$*

$$\mathbb{E}\Big[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^p\Big] \le c_p \Big(\frac{n}{k}\Big)^{p/2}.$$

Moreover, there exists constant c such that we have the following tail bound for all, $t \ge 1$

$$\mathbb{P}\Big(\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 \ge t\frac{cn}{k}\Big) \le e^{-t}.$$

The proof is provided later in Section V-A. Thus, utilizing Lemma 2 and (28), we have obtain the following bound for some constant c_1 and c_2:

$$\mathbb{E}[T] \le \|P - \mathsf{F}\|_{\mathrm{F}}^2 + c_2\frac{n}{k} + 2c_1\sqrt{\frac{n}{k}}\|P - \mathsf{F}\|_{\mathrm{F}}.$$

Let $\mathbb{E}_{H_0}[\cdot]$ and $\mathbb{E}_{H_1}[\cdot]$ denote the expectation operators under hypotheses H_0 and H_1, respectively. In essence, we have established the following bounds on $\mathbb{E}_{H_0}[T]$:

$$|\mathbb{E}_{H_0}[T]| \le c_2\frac{n}{k}, \tag{29}$$

In a similar manner, we can obtain complementary lower bounds on $\mathbb{E}_{H_1}[T]$ (cf. (28)). Consequently, we have the following lower bound on $\mathbb{E}_{H_1}[T]$:

$$\mathbb{E}_{H_1}[T] \ge \|P - \mathsf{F}\|_{\mathrm{F}}^2 - 2c_1\sqrt{\frac{n}{k}}\|P - \mathsf{F}\|_{\mathrm{F}}. \tag{30}$$

Bounding variance. Now we will find bounds on $\mathrm{var}(T)$ under the two hypotheses. For this, we will make use of the law of total variance by conditioning T with respect to $\mathcal{Z}_1$ as

$$\mathrm{var}(T) = \mathbb{E}[\mathrm{var}(T \mid \mathcal{Z}_1)] + \mathrm{var}(\mathbb{E}[T \mid \mathcal{Z}_1]). \tag{31}$$

First, we will examine the term $\mathbb{E}[\text{var}(T|\mathcal{Z}_1)]$. Note that conditioned on $\mathcal{Z}_1$, the term $F(\hat{w}_i - \hat{w}_j)^2$ is constant and does not contribute to $\text{var}(T|\mathcal{Z}_1)$. Moreover, Z_{ij} for $(i,j) \in \mathcal{E}$ are mutually independent, and hence, we can analytically find the expression for $\text{var}(T|\mathcal{Z}_1)$ as

$$\begin{aligned}
\text{var}(T \mid \mathcal{Z}_1) &\overset{\zeta_1}{=} \sum_{(i,j)\in\mathcal{E}} \text{var}\left(\frac{Z_{ij}(Z_{ij}-1)}{k'_{ij}(k'_{ij}-1)}\right) + 4F(\hat{w}_i - \hat{w}_j)^2 \text{var}\left(\frac{Z_{ij}}{k_{ij}}\right) \\
&\qquad - 4F(\hat{w}_i - \hat{w}_j)\left(\frac{\mathbb{E}[Z_{ij}^2(Z_{ij}-1)]}{(k'_{ij})^2(k'_{ij}-1)} - \frac{\mathbb{E}[Z_{ij}(Z_{ij}-1)]}{k'_{ij}(k'_{ij}-1)}\frac{\mathbb{E}[Z_{ij}]}{k'_{ij}}\right) \\
&\overset{\zeta_2}{=} \sum_{(i,j)\in\mathcal{E}} \frac{2p_{ij}^2 + 4(k'_{ij}-2)p_{ij}^3 + (6-4k'_{ij})p_{ij}^4}{k'_{ij}(k'_{ij}-1)} + \frac{4F^2(\hat{w}_i - \hat{w}_j)p_{ij}(1-p_{ij})}{k'_{ij}} \\
&\qquad - 4F(\hat{w}_i - \hat{w}_j)\frac{2(p_{ij}^2 - p_{ij}^3)}{k'_{ij}},
\end{aligned}$$

where ζ_1 follows from the variance of sum technique and ζ_2 follows from the expressions for the first four moments of Binomial random variables and some basic algebra. Now, in order to bound $\mathbb{E}[\text{var}(T \mid \mathcal{Z}_1)]$, we will substitute all $k'_{ij} = k$ for all $(i,j) \in \mathcal{E}$ and simplify the above expression as:

$$\begin{aligned}
\mathbb{E}[\text{var}(T \mid z_1)] &= \sum_{(i,j)\in\mathcal{E}} \frac{2p_{ij}^2 - 4p_{ij}^3 + 2p_{ij}^4}{k(k-1)} \\
&\qquad + p_{ij}(1-p_{ij})\left(\frac{4p_{ij}^2}{k} + \frac{4\mathbb{E}[F^2(\hat{w}_i - \hat{w}_j)]}{k} - \frac{8\mathbb{E}[F(\hat{w}_i - \hat{w}_j)]p_{ij}}{k}\right) \\
&= \sum_{(i,j)\in\mathcal{E}} \frac{2p_{ij}^2(1-p_{ij})^2}{k(k-1)} + \frac{4p_{ij}(1-p_{ij})}{k}(p_{ij} - \mathbb{E}[F(\hat{w}_i - \hat{w}_j)])^2 \\
&\le \frac{nd_{\max}}{8k(k-1)} + \frac{1}{k}\|P - \mathbb{E}[\hat{\mathsf{F}}]\|_{\mathrm{F}}^2 \\
&\le \frac{nd_{\max}}{8k(k-1)} + \frac{1}{k}(\|P - \mathsf{F}\|_{\mathrm{F}} + \|\mathsf{F} - \mathbb{E}[\hat{\mathsf{F}}]\|_{\mathrm{F}})^2 \\
&\le \frac{nd_{\max}}{8k(k-1)} + \frac{1}{k}(\|P - \mathsf{F}\|_{\mathrm{F}} + \mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}])^2 \\
&\le \frac{nd_{\max}}{8k(k-1)} + \frac{1}{k}\left(\|P - \mathsf{F}\|_{\mathrm{F}} + c_1\sqrt{\frac{n}{k}}\right)^2, \qquad (32)
\end{aligned}$$

where the last inequality follows from Lemma 2. Now we will bound the second term of (31), i.e., $\text{var}(\mathbb{E}[T|\mathcal{Z}_1])$. Recall that by (27), we have $\mathbb{E}[T|\mathcal{Z}_1] = \sum_{(i,j)\in\mathcal{E}}(p_{ij} - F(\hat{w}_i - \hat{w}_j))^2$. Therefore, we upper bound $\text{var}(\mathbb{E}[T \mid \mathcal{Z}_1])$ as

$$\begin{aligned}
\text{var}(\mathbb{E}[T \mid \mathcal{Z}_1]) &= \text{var}\left(\sum_{(i,j)\in\mathcal{E}} (p_{ij} - F(\hat{w}_j - \hat{w}_i))^2\right) = \text{var}(\|P - \hat{\mathsf{F}}\|_{\mathrm{F}}^2) \\
&= \mathbb{E}[\|P - \hat{\mathsf{F}}\|_{\mathrm{F}}^4] - \mathbb{E}[\|P - \hat{\mathsf{F}}\|_{\mathrm{F}}^2]^2 \\
&\le \mathbb{E}[(\|P - \mathsf{F}\|_{\mathrm{F}} + \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}})^4] - \mathbb{E}[(\|P - \mathsf{F}\|_{\mathrm{F}} - \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}})^2]^2,
\end{aligned}$$

where the last inequality follows from triangle inequality in the first term and reverse triangle inequality on the second term. Under hypothesis H_0 the above expression simplifies trivially as

$$\text{var}_{H_0}(\mathbb{E}[T \mid \mathcal{Z}_1]) \le c_4\left(\frac{n}{k}\right)^2, \qquad (33)$$

where $\text{var}_{H_l}(\cdot)$ denotes the variance under hypothesis l for $l \in \{0, 1\}$. Now, we turn our attention to bounding $\text{var}_{H_1}(\mathbb{E}[T \mid \mathcal{Z}_1])$. This bound can be established through a relatively mechanical process described as follows

$$\begin{aligned}
\text{var}_{H_1}(\mathbb{E}[T|\mathcal{Z}_1]) &\le \|P - \mathsf{F}\|_{\mathrm{F}}^4 + 4\|P - \mathsf{F}\|_{\mathrm{F}}^3\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}] + 6\|P - \mathsf{F}\|_{\mathrm{F}}^2\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2] \\
&\qquad + 4\|P - \mathsf{F}\|_{\mathrm{F}}\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^3] + \mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^4] \\
&\qquad - (\|P - \mathsf{F}\|_{\mathrm{F}}^2 + \mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2] - 2\|P - \mathsf{F}\|_{\mathrm{F}}\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}])^2
\end{aligned}$$

$$
\begin{aligned}
&= 8\|P-\mathsf{F}\|_{\mathrm{F}}^3\mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}] + 4\|P-\mathsf{F}\|_{\mathrm{F}}^2(\mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}^2] - \mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}]^2) \\
&\qquad + 4\|P-\mathsf{F}\|_{\mathrm{F}}(\mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}^3] + \mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}^2]\mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}]) \\
&\qquad + \mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}^4] - \mathbb{E}[\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}^2]^2 \\
&\le 8c_1\|P-\mathsf{F}\|_{\mathrm{F}}^3\sqrt{\frac{n}{k}} + 4c_2\|P-\mathsf{F}\|_{\mathrm{F}}^2\frac{n}{k} \\
&\qquad\qquad 4(c_3+c_2c_1)\|P-\mathsf{F}\|_{\mathrm{F}}\left(\frac{n}{k}\right)^{3/2} + c_4\left(\frac{n}{k}\right)^2 \qquad (34)
\end{aligned}
$$

Thus, by combining Eqs. (32) and (33) and (34) we obtain the following bounds on $\mathrm{var}_{H_0}(T)$ and $\mathrm{var}_{H_1}(T)$ based on (31)

$$
\mathrm{var}_{H_0}(\mathbb{E}[T]) \le \frac{nd_{\max}}{8k(k-1)} + c_1^2\frac{n}{k^2} + c_4\left(\frac{n}{k}\right)^2 \qquad (35)
$$

$$
\begin{aligned}
\mathrm{var}_{H_1}(\mathbb{E}[T]) &\le \frac{nd_{\max}}{8k(k-1)} + \frac{1}{k}\left(\|P-\mathsf{F}\|_{\mathrm{F}} + c_1\sqrt{\frac{n}{k}}\right)^2 + 8c_1\|P-\mathsf{F}\|_{\mathrm{F}}^3\sqrt{\frac{n}{k}} \\
&\quad + 4c_2\|P-\mathsf{F}\|_{\mathrm{F}}^2\frac{n}{k} + 4\tilde{c}_3\|P-\mathsf{F}\|_{\mathrm{F}}\left(\frac{n}{k}\right)^{3/2} + c_4\left(\frac{n}{k}\right)^2. \qquad (36)
\end{aligned}
$$

We define the decision rule as follows: Select hypothesis H_1 if the test statistic T exceeds the threshold $\tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k}$, i.e.,

$$
T > \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k}, \qquad (37)
$$

where $\tilde{\gamma}$ is a suitably chosen constant (selected below). Consequently, we can employ the one-sided Chebyshev's inequality to bound the probability of error under hypothesis H_0, yielding:

$$
\begin{aligned}
\mathbb{P}_{H_0}\left(T > \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k}\right) &= \mathbb{P}_{H_0}\left(T - \mathbb{E}_{H_0}[T] > \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k} - \mathbb{E}_{H_0}[T]\right) \\
&\le \mathbb{P}_{H_0}(T - \mathbb{E}_{H_0}[T] > \tilde{\gamma}\frac{n}{k}) \\
&\le \frac{\mathrm{var}_{H_0}(T)}{\mathrm{var}_{H_0}(T) + \tilde{\gamma}^2(\frac{n}{k})^2} \\
&\le \frac{\frac{nd_{\max}}{4k^2} + c_1^2\frac{n}{k^2} + c_4(\frac{n}{k})^2}{\frac{nd_{\max}}{4k^2} + c_1^2\frac{n}{k^2} + c_4(\frac{n}{k})^2 + \tilde{\gamma}^2(\frac{n}{k})^2} \\
&= \frac{\frac{d_{\max}}{4n} + c_1^2\frac{1}{n} + c_4}{\frac{d_{\max}}{4n} + c_1^2\frac{1}{n} + c_4 + \tilde{\gamma}^2} \le \frac{1}{4},
\end{aligned}
$$

where the last bound holds for an appropriate constant such as $\tilde{\gamma} \ge \max\{4c_4, 4, 4c_1/\sqrt{n}\}$. This is because of the fact that $d_{\max} \le n$.

Now, we will find an upper bound on the probability of error under hypothesis H_1. Observe that an error is made under H_1 if the value of the test statistic $T \le \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k}$.

$$
\begin{aligned}
&\mathbb{P}_{H_1}\left(T \le \tilde{\gamma}\frac{n}{k} + c_3^2\frac{n}{k}\right) \\
&\qquad = \mathbb{P}_{H_1}\left(T - \mathbb{E}_{H_1}[T] \le \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k} - \mathbb{E}_{H_1}[T]\right) \\
&\qquad \overset{\zeta_1}{\le} \mathbb{P}\left(T - \mathbb{E}_{H_1}[T] \le \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k} + 2c_1\sqrt{\frac{n}{k}}\|P-\mathsf{F}\|_{\mathrm{F}} - \|P-\mathsf{F}\|_{\mathrm{F}}^2\right) \\
&\qquad \overset{\zeta_2}{\le} \frac{\mathrm{var}_{H_1}(T)}{\mathrm{var}_{H_1}(T) + (D^2-\Delta)^2} \overset{\zeta_3}{\le} \frac{1}{4},
\end{aligned}
$$

where ζ_1 follows from (30), ζ_2 follows by one-sided Chebyshev inequality and defining $D = \|P-\mathsf{F}\|_{\mathrm{F}}$ and $\Delta = \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k} + 2c_1\sqrt{\frac{n}{k}}D$. The step ζ_3 holds if $4\mathrm{var}_{H_1}(T) \le (D^2-\Delta)^2$ or equivalently if $D^2 \ge 2\sqrt{\mathrm{var}_{H_1}(T)} + \Delta$. Using the sub-additivity of $\sqrt{\cdot}$ operator, the following condition necessitates that for this to be true:

$$
D^2 \ge 2\left(\frac{\sqrt{nd_{\max}}}{2k} + \frac{1}{\sqrt{k}}\left(D + c_1\sqrt{\frac{n}{k}}\right) + 2\sqrt{2c_1}D^{3/2}\left(\frac{n}{k}\right)^{\frac{1}{4}} + 2\sqrt{c_2}D\sqrt{\frac{n}{k}}\right.
$$

$$+ 2\sqrt{\tilde{c}_3}\sqrt{D}\Big(\frac{n}{k}\Big)^{\frac{3}{4}} + c_4\frac{n}{k}\Big) + \tilde{\gamma}\frac{n}{k} + c_2\frac{n}{k} + 2c_1\sqrt{\frac{n}{k}}D.$$

Substituting $D = a_0\sqrt{\frac{n}{k}}$ in the above expression, for some constant a_0, we obtain:

$$a_0^2 \geq \frac{2}{\sqrt{2nd_{\max}}} + 2(a_0 + c_1)\sqrt{\frac{1}{n}} + 4\sqrt{2c_1}a_0^{3/2} + 4\sqrt{c_2}a_0 + 4\sqrt{2\tilde{c}_3 a_0} + \tilde{\gamma} + c_2 + 2a_0c_1.$$

Again utilizing the fact that $\lambda_2(L) \leq 2d_{\max}$, we can conclude that the above expression is true for some large enough constant a_0 independent of n and k, thus establishing that for $D = a_0\sqrt{\frac{n}{k}}$ and a_0 large enough our decision rule achieves a type I and type II sum error of at most $1/2$. Utilizing Theorem 1 we obtain that $\inf_{w\in\mathcal{W}_b} \|P - \mathsf{F}(w)\|_{\mathrm{F}} = \Theta(\|P - \mathsf{F}\|_{\mathrm{F}})$. Combining this fact along with the definition of critical threshold (cf. (14)), we have the following bound on ε_{c}:

$$\varepsilon_{\mathsf{c}} \leq O\Bigg(\sqrt{\frac{1}{nk}}\Bigg).$$

This completes the proof. □

Remarks. Notably, from the above result, we have that $\varepsilon_{\mathsf{c}} \to 0$ as n or k goes to infinity. Therefore, for any fixed n and $\epsilon > 0$, our decision rule is guaranteed to achieve a non-trivial minimax risk (strictly less than 1) for any pairwise comparison model P in the class $\mathcal{M}_0$ or $\mathcal{M}_1(\epsilon)$ as long as the number of observed samples for each pair (i.e. k) are sufficiently large. Moreover, there do exist pairwise comparison models $\{p_{ij} : (i,j) \in \mathcal{E}\}$ whose (normalized) separation is constant with n. Consequently, for such models, we can argue that for any fixed k and n large enough, our decision rule will achieve a non-trivial minimax risk. One such example of a pairwise comparison model represented by its pairwise comparison matrix (on a complete graph) is

$$P = \Big(\frac{1}{2} + \eta\Big)(\mathbf{1}\mathbf{1}^{\mathrm{T}} - I), \quad \text{for any } \eta \in \Big(0, \frac{1}{2}\Big).$$

It is easy to verify that for this comparison model, we must have $\inf_{w\in\mathcal{W}_b} \frac{1}{n}\|P - \mathsf{F}(w)\|_{\mathrm{F}} \geq \eta$. This is because any matrix $\mathsf{F}(w)$ must satisfy the constraint $(\mathsf{F})_{ij} + (\mathsf{F})_{ji} = 1$ for every $i \neq j$, which immediately leads to the lower bound of η on the separation distance.

A. *Proof of Lemma 2*

Observe that by definition of F and $\hat{\mathsf{F}}$, we have

$$\begin{aligned}\sum_{(i,j)\in\mathcal{E}} (F(w_i^* - w_j^*) - F(\hat{w}_i - \hat{w}_j))^2 &\leq \beta^2 \sum_{(i,j)\in\mathcal{E}} (|(w_i^* - w_j^*) - (\hat{w}_i - \hat{w}_j)|)^2 \\ &\leq 4\beta^2\|\hat{w} - w^*\|_L^2. \end{aligned} \tag{38}$$

Taking the power $p/2$ on both sides and then taking the expectation, we obtain

$$\mathbb{E}[\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^p] \leq 2^p\beta^p\mathbb{E}[\|\hat{w} - w^*\|_L^p] \leq c_p\Big(\frac{n}{k}\Big)^{p/2},$$

where the last inequality follows by plugging in the bounds on pth moment from Theorem 3 and absorbing the constant α, β in c_p. The tail bound follows directly from (38). □

VI. Proofs of Theorem 4 and Proposition 2

A. *Proof of Theorem 4*

Without loss of generality, we assume that the graph $\mathcal{G}$ is Eulerian. If not, we can reduce the problem to an Eulerian graph by considering the largest Eulerian-spanning subgraph $\tilde{\mathcal{G}}$ of $\mathcal{G}$ so that every vertex of $\tilde{\mathcal{G}}$ has even degree, which exists by our assumption that $\mathcal{G}$ is super-Eulerian.

Under the null hypothesis, we assume that the pairwise comparison model P is a uniform distribution, i.e., $p_{ij} = \frac{1}{2}, \forall\ (i,j) \in \mathcal{E}$ and let $\mathbb{P}_0$ denote the probability measure corresponding to this pairwise comparison model P. Under the alternative hypothesis, we will set our pairwise comparison model R to be a perturbed version of the uniform distribution (sharing the same observation graph $\mathcal{G}$). Specifically, every perturbation will have the following property:

$$\forall (i,j) \in \mathcal{E},\ r_{ij} = \frac{1}{2} + \eta b_{ij},\ \text{where } \eta \in \big[0, \frac{1}{2} - \delta\big],\ b_{ij} \in \{-1, 1\},\ b_{ij} + b_{ji} = 0,\ \forall i \in [n],$$

$$\text{and} \sum_{j:(i,j)\in\mathcal{E}} b_{ij} = 0, \forall i \in [n], \tag{39}$$

where b_{ij} represents the signs of the perturbation by parameter η. Note that we set $b_{ij} = 0$ for $(i,j) \notin \mathcal{E}$. Let any such sequence of perturbations b_{ij} is represented by a matrix $B \in \{-1,0,1\}^{n\times n}$.

As we delve deeper into the perturbation structure, we will carefully select a subset of perturbations $\mathcal{B}$ satisfying the constraints in (39), as well as additional constraints to be specified later

$$\mathcal{B} \subseteq \left\{ b_{ij} \in \{-1,1\} \text{ for } (i,j) \in \mathcal{E} : b_{ij} + b_{ji} = 0, \sum_{j:(i,j)\in\mathcal{E}} b_{ij} = 0, \forall i \in [n] \right\}. \tag{40}$$

Based on this selection, under the alternative hypothesis, let the pairwise comparison model R be generated from a mixture distribution such that $R = P + \eta B$ and $B \sim \mathrm{Unif}(\mathcal{B})$, i.e., R is generated by adding the perturbation sequence selected uniformly at random from $\mathcal{B}$. Let $\mathbb{P}_{\mathcal{B}}$ denote the measure corresponding to the overall mixture distribution.

As we examine the perturbation structure, we make our first observation: for any perturbation B satisfying (39), the corresponding pairwise comparison model R belongs to the class $\mathcal{M}_1(\epsilon)$, for some ϵ as a function of η. Specifically, we will show that the perturbation B guarantees a minimum separation distance of ϵ from the class of $\mathcal{M}_0$.

Bounding separation. Our first observation is that any such perturbation $R = P + \eta B$ has a sufficiently large and (more importantly a tractable) separation distance. In order to lower bound this separation distance we will utilize Theorem 1. But first, we need to find the optimal $\mathcal{T}_F$ weights w^* (as in (8)). This is addressed in the following lemma.

Lemma 3 (Optimal Weights for Perturbed Matrix)**.** *For the $\mathcal{T}_F$ model and for any $B \in \mathcal{B}$ defined as in (40), the perturbed pairwise comparison matrix $P + \eta B$ has a unique optimal $\mathcal{T}_F$ weights given by $w^* = \mathbf{0}$ (all zeros vector).*

The proof is provided later in Section VI-C. We now utilize Theorem 1 to obtain the following lower bound on separation distance as

$$\begin{aligned}\inf_{w\in\mathcal{W}_b} \|P + \eta B - \mathsf{F}(w)\|_{\mathrm{F}} &\geq c_1 \|P + \eta B - \mathsf{F}(\mathbf{0})\|_{\mathrm{F}} \\ &= c_1 \eta \sqrt{|\mathcal{E}|}\end{aligned}$$

where c_1 is the lower bound constant in Theorem 1 and the last equality follows sinde $(P)_{ij} = (\mathsf{F}(0))_{ij} = 1/2$ for all $(i,j) \in \mathcal{E}$. Therefore, we have $n\epsilon \geq c_1\eta\sqrt{|\mathcal{E}|}$ by definition of $\mathcal{M}_1(\epsilon)$.

Having established a lower bound on the separation distance for each of the perturbations, our next step is to carefully select a special subset $\mathcal{B}$ of perturbations that allows us to approximate the "degrees of freedom" in the structure of our perturbation set, while also taking into account the constraints imposed by the graph topology.

To this end, we leverage the assumption that our observation graph $\mathcal{G}$ is Eulerian, meaning every node has an even degree. This property enables us to decompose $\mathcal{G}$ into a collection of edge-disjoint cycles, denoted by $\mathcal{C}$. In addition, we introduce a comparison incidence graph $\mathcal{G}_I$, which represents the comparison structure as an undirected bipartite graph. This graph has n item nodes on one side and $|\mathcal{E}|/2$ nodes on the other side, each representing a pairwise comparison $(i,j) \in \mathcal{E}$ for $j > i$. The edges in $\mathcal{G}_I$ connect items to their respective comparison nodes. Since every node in $\mathcal{G}$ has an even degree, this ensures that the incidence graph $\mathcal{G}_I$ is Eulerian, and therefore $\mathcal{G}_I$ also has a cycle decomposition denoted by $\mathcal{C}_I$. Notably, each cycle in $\mathcal{G}_I$ is of even length and we can establish a one-to-one correspondence between the cycles in $\mathcal{C}$ and $\mathcal{C}_I$. Now, we orient the edges in the undirected comparison incidence graph $\mathcal{G}_I$ based on the values b_{ij} in the perturbation B. Specifically, we will orient the edges in $\mathcal{G}_I$ as follows: if $b_{ij} = 1$, the edge will point from the item node to the comparison node for pair (i,j), and if $b_{ij} = -1$, the edge will have the opposite direction. The constraints in (40) ensure that each node in $\mathcal{G}_I$ has equal in-degree and out-degree.

To specify the construction of $\mathcal{B}$, we consider any fixed cycle decomposition $\mathcal{C}_I$ (since it may not be unique). Let the number of cycles in the cycle decomposition be denoted by $|\mathcal{C}_I|$. Let $\sigma_i \in \mathcal{C}_I$ represents the ith cycle in $\mathcal{C}_I$ and $|\sigma_i|$ denotes the length of this ith cycle. Observe that, we can independently orient the edges of any cycle $\sigma_i \in \mathcal{C}_I$ in either clockwise or counterclockwise direction yielding $2^{|\mathcal{C}_I|}$ distinct Eulerian orientations for $\mathcal{G}_I$. We then construct the structured collection of perturbations $\mathcal{B}$ by associating each perturbation with one of the $2^{|\mathcal{C}_I|}$ distinct Eulerian orientations of the cycle decomposition $\mathcal{C}_I$. This establishes a one-to-one correspondence between valid perturbation in $\mathcal{B}$ and distinct Eulerian orientations of $\mathcal{C}_I$. Thus, in summary we define $\mathcal{B}$ correspoding to decomposition $\mathcal{C}_I$ as

$$\begin{aligned}\mathcal{B} \triangleq \Bigg\{ b_{ij} \in \{-1,1\} : b_{ij} + b_{ji} = 0, \forall (i,j) \in \mathcal{E}, &\sum_{j:(i,j)\in\mathcal{E}} b_{ij} = 0, \forall i \in [n], \\ &|b_l - b_{l+1}| = 2, \forall l \in \sigma_i, \forall \sigma_i \in \mathcal{C}_I \Bigg\},\end{aligned}$$

where, l is used for indexing sequential edges of the cycle σ_i.

Bounding risk. Now, we will utilize the Ingster-Suslina method to compute lower bound on $\mathcal{R}(\mathcal{G},\epsilon)$ (cf. (13)). The standard testing inequality by Le-Cam [46] states that

$$\mathcal{R}(\mathcal{G},\epsilon) \geq 1 - \|\mathbb{P}_0 - \mathbb{P}_{\mathcal{B}}\|_{\mathsf{TV}} \geq 1 - \sqrt{\chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0)}. \tag{41}$$

We calculate the chi-squared divergence $\chi^2(\mathbb{P}_0||\mathbb{P}_{\mathcal{B}})$ by expressing it as an expectation with respect to two independent pairwise models corresponding to permutation B and B' drawn independently and uniformly at random from $\mathcal{B}$ as

$$\chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0) = \mathbb{E}_{B,B'\sim\mathsf{Unif}(\mathcal{B})}\left[\int \frac{d\mathbb{P}_B d\mathbb{P}_{B'}}{d\mathbb{P}_0}\right].$$

We will now leverage the tensorization property of $1+\chi^2(P||Q)$ which enables us to decompose the chi-squared divergence between product distributions into a product of individual divergences. Specifically, for distributions $P_1, Q_1, \ldots, P_n, Q_n$, we have

$$1 + \chi^2\left(\prod_{i=1}^n P_i \middle|\middle| \prod_{i=1}^n Q_i\right) = \prod_{i=1}^n (1 + \chi^2(P_i||Q_i)).$$

Consequently, the χ^2-divergence $\chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0)$ simplifies as

$$\begin{aligned}
&1 + \chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0) = \\
&\mathbb{E}_{B,B'\sim\mathsf{Unif}(\mathcal{B})}\Bigg[\prod_{(i,j)\in\mathcal{E}}\Bigg(\sum_{m=0}^k \frac{\binom{k}{m}(\frac{1}{2}+\eta b_{ij})^m(\frac{1}{2}-\eta b_{ij})^{k-m}\binom{k}{m}(\frac{1}{2}+\eta b'_{ij})^m(\frac{1}{2}-\eta b'_{ij})^{k-m}}{\binom{k}{m}\left(\frac{1}{2}\right)^k}\Bigg)\Bigg] \\
&= \mathbb{E}_{B,B'\sim\mathsf{Unif}(\mathcal{B})}\Bigg[\prod_{(i,j)\in\mathcal{E}}\Bigg(\sum_{m=0}^k \frac{\binom{k}{m}(\frac{1}{2}+\eta b_{ij})^m(\frac{1}{2}-\eta b_{ij})^{k-m}(\frac{1}{2}+\eta b'_{ij})^m(\frac{1}{2}-\eta b'_{ij})^{k-m}}{\left(\frac{1}{2}\right)^k}\Bigg)\Bigg].
\end{aligned} \tag{42}$$

We direct our attention to the (i,j)th term of the product in (42), for $(i,j)\in\mathcal{E}$ and denote it as $h(b_{ij}, b'_{ij})$

$$h(b_{ij}, b'_{ij}) = \sum_{m=0}^k \frac{\binom{k}{m}(\frac{1}{2}+\eta b_{ij})^m(\frac{1}{2}-\eta b_{ij})^{k-m}(\frac{1}{2}+\eta b'_{ij})^m(\frac{1}{2}-\eta b'_{ij})^{k-m}}{\left(\frac{1}{2}\right)^k}. \tag{43}$$

Now since we have $b_{ij}, b'_{ij} \in \{-1, 1\}$, therefore whenever b_{ij} and b'_{ij} agree, by (43) we have $h(1,1) = h(-1,-1)$. And moreover, we can calculate $h(1,1)$ as

$$\begin{aligned}
h(1,1) &= 2^k \sum_{m=0}^k \binom{k}{m}\left(\frac{1}{2}+\eta\right)^{2m}\left(\frac{1}{2}-\eta\right)^{2k-2m} \\
&= 2^k \sum_{m=0}^k \binom{k}{m}\left(\frac{1}{4}+\eta^2+\eta\right)^{m}\left(\frac{1}{4}+\eta^2-\eta\right)^{k-m} \\
&= \left(1+4\eta^2\right)^k \sum_{m=0}^k \binom{k}{m}\left(\frac{1}{2}+\frac{\eta}{\frac{1}{2}+2\eta^2}\right)^{m}\left(\frac{1}{2}-\frac{\eta}{\frac{1}{2}+2\eta^2}\right)^{k-m} \\
&= (1+4\eta^2)^k.
\end{aligned}$$

Additionally, by (43) we also have $h(1,-1) = h(-1,1)$ and it simplifies to

$$h(1,-1) = 2^k \sum_{m=0}^k \binom{k}{m}\left(\frac{1}{2}+\eta\right)^{2k}\left(\frac{1}{2}-\eta\right)^{2k} = (1-4\eta^2)^k.$$

For any two perturbations $B, B' \sim \mathsf{Unif}(\mathcal{B})$, let random variables A_1 denotes the number of agreements between B, B' respectively, i.e., number of $(i,j)\in\mathcal{E}_+$ where $b_{ij} = b'_{ij}$ in randomly drawn permutation B and B'. And similarly let A_2 denotes the number of disagreements between B, B' i.e., number of $(i,j)\in\mathcal{E}_+$ where $b_{ij} = -b'_{ij}$. Consequently, we obtain

$$1 + \chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0) = \mathbb{E}_{B,B'\sim\mathsf{Unif}(\mathcal{B})}\left[h(1,1)^{2A_1} h(1,-1)^{2A_2}\right]$$

$$= \mathbb{E}_{B,B'\sim \text{Unif}(\mathcal{B})}\left[(1+4\eta^2)^{2kA_1}(1-4\eta^2)^{2kA_2}\right]$$
$$\leq \mathbb{E}_{B,B'\sim \text{Unif}(\mathcal{B})}\left[\exp\big(8\eta^2 k(A_1-A_2)\big)\right]. \tag{44}$$

In addition, we define vectors $a, a' \in \{-1,1\}^{|\mathcal{C}_I|}$ to represent the orientations of the $|\mathcal{C}_I|$ cycles in $\mathcal{G}_I$ induced by B. The subsequent calculation will now be used to complete the proof:

$$\chi^2(\mathbb{P}_{\mathcal{B}}||\mathbb{P}_0)+1 \overset{\zeta_1}{\leq} \frac{1}{2^{2|\mathcal{C}_I|}}\sum_{B,B'}\exp\big(8\eta^2 k(A_1-A_2)\big) \overset{\zeta_2}{=} \frac{1}{2^{2|\mathcal{C}_I|}}\sum_{a,a'}\exp\left(8\eta^2 k\sum_{\sigma_i\in\mathcal{C}_I}|\sigma_i|a_i a_i'\right)$$
$$= \mathbb{E}\left[\prod_{\sigma_i\in\mathcal{C}_I}\exp\big(8\eta^2 k|\sigma_i|a_i a_i'\big)\right] \overset{\zeta_3}{=} \prod_{\sigma_i\in\mathcal{C}_I}\mathbb{E}\left[\exp\big(8\eta^2 k|\sigma_i|a_i a_i'\big)\right]$$
$$= \prod_{\sigma_i\in\mathcal{C}_I}\left(\frac{1}{2}\exp\big(8\eta^2 k|\sigma_i|\big)+\frac{1}{2}\exp\big(-8\eta^2 k|\sigma_i|\big)\right)$$
$$\leq \prod_{\sigma_i\in\mathcal{C}_I}\left(\exp\Big(32\eta^4 k^2(|\sigma_i|)^2\Big)\right) = \exp\left(32\eta^4 k^2\sum_{\sigma_i\in\mathcal{C}_I}|\sigma_i|^2\right),$$

where ζ_1 follows from (44) and the fact that there are $2^{|\mathcal{C}_I|}$ perturbations which are sampled uniformly from $\mathcal{B}$, ζ_2 follows from definition of a_i and the fact that number of agreements/disagreements can be represented in terms of the agreements/disagreements of the cycle orientations a_i, a_i'. ζ_3 follows from the fact that the orientations of the cycles are independent of one another. Finally, utilizing the fact that $c_1\eta\sqrt{|\mathcal{E}|} \leq n\epsilon$ and by combining the resulting bound along with (41) and the fact that cycle lengths in $\mathcal{G}_I$ are twice the size in $\mathcal{G}$ completes the proof. □

B. Proof of Proposition 2

Part 1. For a complete graph, the comparison incidence graph $\mathcal{G}_I$ has n item nodes and $\frac{n(n-1)}{2}$ comparison nodes. When n is odd, all nodes have an even degree equal to $n-1$, therefore $\mathcal{G}$ is Eulerian. Notably, n can take the forms $n = 6m+1$, $n = 6m+3$, and $n = 6m+5$, where $m \in \mathbb{N}$. As established by Kirkman [51], for $n = 6m+1$ and $n = 6m+3$, $\mathcal{G}$ can always be decomposed into cycles of length 3. Meanwhile, for $n = 6x+5$, $\mathcal{G}$ can be decomposed into a cycle of length 4 and remaining cycles of length 3 [52]. Therefore, we have $|\mathcal{E}|^2 = O(n^4)$ and $\sum_{\sigma\in\mathcal{C}}|\sigma|^2 = O(n^2)$, giving $\varepsilon_{\mathsf{c}}^2 = \Omega(1/nk)$

Part 2. For graphs comprising a single cycle, it is easy to verify that the number of cycles is 1 and the cycle has a length of n.

Part 3. Consider a d-regular graph with even degree d. The associated comparison incidence graph has n item nodes and $nd/2$ comparison nodes. By applying [40, Lemma 9], we can decompose the edge set of the comparison incidence graph into cycles of size at most $\lfloor 2\log_2(n)\rfloor$, with at most $\min\{2n+nd, 4n\} = 4n$ edges remaining. Since the graph is Eulerian, removing cycles does not affect this property. Therefore, the remaining $4n$ edges can be further decomposed into cycles of length at most $2n$ (since cycles can have a maximum length of $2n$ and this reflects a worst-case scenario). This yields $\sum_{\sigma\in\mathcal{C}}|\sigma|^2 = O(n^2)$, which in turn implies $\varepsilon_{\mathsf{c}}^2 = \Omega(1/n^2k)$.

Part 4. For a toroidal grid of size $\sqrt{n}\times\sqrt{n}$, we can generate a cycle decompostion of $\mathcal{G}$ consisting of $\sqrt{n}$ horizontal edges and $\sqrt{n}$ vertical edges. Clearly, each of these edges have a length of $\sqrt{n}$. Therefore, $\sum_{\sigma_i\in\mathcal{C}}|\sigma_i|^2 = 2n\sqrt{n}$. And since it is a toroidal grid we have $|\mathcal{E}| = 2n$. Plugging in these values we obtain $\varepsilon_{\mathsf{c}}^2 = \Omega(1/n^{7/4}k)$. □

C. Proof of Lemma 3

In order to find the optimal weights w^*, our objective is to solve the following optimization problem with parameter b :

$$l^*(w) = \min_{w\in\mathcal{W}_b} -\sum_{(i,j)\in\mathcal{E}} r_{ij}\log(F(w_i-w_j)) + (1-r_{ij})\log(1-F(w_i-w_j)).$$

Our initial observation is that, due to the skew-symmetrization of the model $r_{ij}+r_{ji}=1$, we can express the gradient of $l^*(w)$ as:

$$(\nabla l^*(w))_i = -2\sum_{j:(i,j)\in\mathcal{E}}(r_{ij}-F(w_i-w_j))\times\frac{F'(w_i-w_j)}{F(w_i-w_j)(1-F(w_i-w_j))}.$$

Furthermore, it is evident that the gradient is zero at $w = \mathbf{0}$. To see this, note that for all $i \in [n]$, we have:

$$(\nabla l^*(w))_i|_{w=\mathbf{0}} = -2 \sum_{j:(i,j)\in\mathcal{E}} (\frac{1}{2} + \eta b_{ij} - F(0)) \times \frac{F'(0)}{F(0)(1-F(0))} = -8\eta F'(0) \sum_{j:(i,j)\in\mathcal{E}} b_{ij} = 0,$$

where the last step is followed by our construction of perturbation sequence in (39). Considering that the gradient is zero at $w = \mathbf{0}$ and the optimization objective is convex over $\mathcal{W}_b$ (in fact strongly convex over $\mathcal{W}_b$), coupled with the uniqueness of the optimal $\mathcal{T}_F$ weights as indicated by Proposition 1, we conclude that $w^* = \mathbf{0}$ is indeed the optimal and unique solution for any $b \geq 0$. □

VII. Proof of Time-Uniform Bounds on Type I and Type II Error Probabilities

In this appendix, we will establish bounds on type I and type II error probabilities as described in Theorem 5. First, we will introduce essential notation to facilitate our analysis and present our proof of Proposition 3. Then, we will establish a few auxiliary lemmata such as which are needed to derive the bounds on type I and type II error probabilities. Finally, combining these results, we will prove Theorem 5 in Section VII-C and a few corollaries based on these results in Appendix A.

Additional notation. We introduce Y_{ij}^l for $l \in \mathbb{N}$ and $(i,j) \in \mathcal{E}$ to denote the observed comparisons that are used for estimating $Z_{ij} = \sum_{l=1}^{k_2} Z_{ij}^{k_1+l}$, i.e., we let $Y_{ij}^l = Z_{ij}^{k_1+l}$ for $l \in [k_2]$. Also, define the statistic $\bar{Y}_{ij}^k \triangleq \sum_{m=1}^k Y_{ij}^m$. Moreover, define $\mathbf{1}_n$ as the all ones vector of length n and I_n as the identity matrix of size $n \times n$.

A. Proof of Proposition 3

We will focus on the following sequence $\{\tilde{T}_{ij}^k, k \in \mathbb{N} \setminus 1\}$ defined as

$$\tilde{T}_{ij}^k \triangleq \frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k - 1)}{k(k-1)} + b_{ij}^2 - 2b_{ij}\frac{\bar{Y}_{ij}^k}{k}.$$

Note that with $b_{ij} = F(\hat{w}_i - \hat{w}_j)$, the term $\tilde{T}_{ij}^k$ reduces to the (i,j)th term of the test-statistic $T^{k_1,k}$ (based on the notation defined above) and we will now show that it is indeed a reverse martingale. To do this, we will demonstrate that both the terms $\frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k-1)}{k(k-1)}$ and $\frac{\bar{Y}_{ij}^k}{k}$ are indeed reverse martingales. First, we focus on the former term. Observe that we can write the product $\bar{Y}_{ij}^k(\bar{Y}_{ij}^k - 1)$ as

$$\frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k - 1)}{k(k-1)} = \frac{(\sum_{m=1}^k Y_{ij}^m)^2 - \sum_{m=1}^k Y_{ij}^m}{k(k-1)} = \frac{1}{k(k-1)} \sum_{l,m=1:l\neq m}^k Y_{ij}^l Y_{ij}^m,$$

where the last equality follows because $\sum_{m=1}^k Y_{ij}^m(Y_{ij}^m - 1) = 0$ as $Y_{ij}^m \in \{0,1\}$. Also, observe that $\mathbb{E}[Y_{ij}^m Y_{ij}^l \mid \mathcal{F}_{k+1}]$ $= \mathbb{E}[Y_{ij}^m Y_{ij}^r \mid \mathcal{F}_{k+1}]$ for $l \neq m \neq r$ and where $\mathcal{F}_k$ is the canonical reverse filtration defined as the σ-algebra generated by $\mathcal{F}_k = \bigotimes_{(i,j)\in\mathcal{E}} \sigma\left(\frac{\bar{Y}_{ij}^k}{k}, Y_{ij}^{k+1}, Y_{ij}^{k+2}, \dots\right)$. This is because for any set $\mathcal{A} \in \mathcal{F}_{k+1}$ and $l, m, r \in [k]$ and $l \neq m \neq r$, we have

$$\mathbb{E}[Y_{ij}^m Y_{ij}^l \mathbb{1}_{\mathcal{A}}] = \mathbb{E}[Y_{ij}^m Y_{ij}^r \mathbb{1}_{\mathcal{A}}] \text{ and } \mathbb{E}[Y_{ij}^m \mathbb{1}_{\mathcal{A}}] = \mathbb{E}[Y_{ij}^l \mathbb{1}_{\mathcal{A}}].$$

Utilizing the above relations, we can show that $\frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k-1)}{k(k-1)}$ is indeed a reverse-martingale as:

$$\begin{aligned}
\mathbb{E}\left[\frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k - 1)}{k(k-1)} \mid \mathcal{F}_{k+1}\right] &= \mathbb{E}\left[\frac{\sum_{l,m=1:l\neq m}^k Y_{ij}^l Y_{ij}^m}{k(k-1)} \mid \mathcal{F}_{k+1}\right] = \frac{\sum_{l,m=1:l\neq m}^{k+1} \mathbb{E}[Y_{ij}^l Y_{ij}^m \mid \mathcal{F}_{k+1}]}{k(k-1)} \\
&= \mathbb{E}\left[\frac{\sum_{l,m=1:l\neq m}^{k+1} Y_{ij}^l Y_{ij}^m}{k(k+1)} \mid \mathcal{F}_{k+1}\right] = \mathbb{E}\left[\frac{\bar{Y}_{ij}^{k+1}(\bar{Y}_{ij}^{k+1} - 1)}{k(k+1)} | \mathcal{F}_{k+1}\right] \\
&= \frac{\bar{Y}_{ij}^{k+1}(\bar{Y}_{ij}^{k+1} - 1)}{k(k+1)},
\end{aligned}$$

where the last equality follows since $\bar{Y}_{ij}^{k+1}$ is measurable with respect to $\mathcal{F}_{k+1}$. Similarly, we can also show that $\frac{\bar{Y}_{ij}^k}{k}$ is also a reverse martingale as:

$$\mathbb{E}\left[\frac{\bar{Y}_{ij}^k}{k} | \mathcal{F}_{k+1}\right] = \frac{\sum_{m=1}^k \mathbb{E}[Y_{ij}^m \mid \mathcal{F}_{k+1}]}{k} = \frac{\sum_{m=1}^k \mathbb{E}[Y_{ij}^m \mid \mathcal{F}_{k+1}]}{k+1}$$

$$= \mathbb{E}\left[\frac{\bar{Y}_{ij}^{k+1}}{k+1}|\mathcal{F}_{k+1}\right] = \frac{\bar{Y}_{ij}^{k+1}}{k}.$$

Finally, the proposition follows by the linearity of conditional expectation and substituting $b_{ij} = F(\hat{w}_i - \hat{w}_j)$ as:

$$\begin{aligned}
\mathbb{E}\big[T^{k_1,k} \mid \mathcal{F}_{k+1}\big] &= \mathbb{E}\left[\sum_{(i,j)\in\mathcal{E}} \tilde{T}_{ij}^{k} \mid \mathcal{F}_{k+1}\right] \\
&= \sum_{(i,j)\in\mathcal{E}} \mathbb{E}\left[\frac{\bar{Y}_{ij}^{k}\big(\bar{Y}_{ij}^{k}-1\big)}{k(k-1)} \mid \mathcal{F}_{k+1}\right] + b_{ij}^2 - 2b_{ij}\mathbb{E}\left[\frac{\bar{Y}_{ij}^{k} \mid \mathcal{F}_{k+1}}{k}\right] \\
&= \sum_{(i,j)\in\mathcal{E}} \frac{\bar{Y}_{ij}^{k+1}\big(\bar{Y}_{ij}^{k+1}-1\big)}{k(k+1)} + b_{ij}^2 - 2b_{ij}\frac{\bar{Y}_{ij}^{k+1}}{k+1} = \sum_{(i,j)\in\mathcal{E}} \tilde{T}_{ij}^{k+1} = T^{k_1,k+1}.
\end{aligned}$$

This completes the proof. □

B. Intermediary Lemmata

In order to derive the proof of Theorem 5, we will first prove the following intermediary lemma that gives bounds on type I and type II error probabilities where the threshold is a function of the estimated weights $\hat{w}$. Additionally, the lemma relies on a variant of the Hanson-Wright inequality that is time-uniform (see Lemma 5) and specialized to our setting; this inequality is also proved in this subsection.

Lemma 4 (Conditional Bounds on Type I and Type II Error Probabilities)**.** *For any $\alpha \in (0,1]$ and for $\hat{w}$ estimated from the first k_1 pairwise comparison for each pair in $\mathcal{E}$, there exist a constant c such that for $\nu \in (0, 1/e)$, the following bounds hold under hypothesis H_0 and H_1 respectively:*

$$\begin{aligned}
&\mathbb{P}_{H_0}\left(\exists k \geq 2, T^{k_1,k} \geq \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 + c\frac{\sqrt{|\mathcal{E}|}}{k}\ell_{k,\nu} + 4\frac{\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}}{\sqrt{k}}\sqrt{\ell_{k,\nu}}\right) \leq \nu, \\
&\mathbb{P}_{H_1}\Bigg(\exists k \geq 2, T^{k_1,k} - \|P - \mathsf{F}\|_{\mathrm{F}}^2 \geq \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 - 2\|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}\|P-\mathsf{F}\|_{\mathrm{F}} - \\
&\qquad\qquad - c\frac{\sqrt{|\mathcal{E}|}}{k}\ell_{k,\nu} - 4\frac{\|P-\mathsf{F}\|_{\mathrm{F}} + \|\mathsf{F}-\hat{\mathsf{F}}\|_{\mathrm{F}}}{\sqrt{k}}\sqrt{\ell_{k,\nu}}\Bigg) \leq \nu,
\end{aligned}$$

where $\ell_{k,\nu} = \log\big(3.5\log^2(k)/\nu\big)$.

Proof.

Part 1. We will first prove the bound under hypothesis H_0. Based on the additional notation defined at the beginning of the Appendix, we can simplify the (i,j)th term $T_{ij}^{k_1,k}$ of the test-statistic $T^{k_1,k}$ as:

$$\begin{aligned}
T_{ij}^{k_1,k} &= \frac{\bar{Y}_{ij}^{k}(\bar{Y}_{ij}^{k}-1)}{k(k-1)} + F(\hat{w}_i - \hat{w}_j)^2 - 2F(\hat{w}_i - \hat{w}_j)\frac{\bar{Y}_{ij}^{k}}{k} \\
&= \frac{\sum_{l,m=1:l\neq m}^{k} Y_{ij}^{m}Y_{ij}^{l}}{k(k-1)} + F(\hat{w}_i - \hat{w}_j)^2 - 2\frac{F(\hat{w}_i - \hat{w}_j)_{l=1}^{m} Y_{ij}^{m}}{k} \\
&\overset{\zeta_1}{=} (y_{ij}^{k})^{\mathrm{T}} A^{(k)} y_{ij}^{k} + \mathbf{1}_k^{\mathrm{T}} A^{(k)}\mathbf{1}_k F(w_i^* - w_j^*)^2 - 2F(w_i^* - w_j^*)\mathbf{1}_k^{\mathrm{T}} A^{(k)} y_{ij}^{k} + \\
&\quad (F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))^2 - 2(F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))\left(\frac{\bar{Y}_{ij}^{k}}{k} - F(w_i^* - w_j^*)\right) \\
&\overset{\zeta_2}{=} \underbrace{(y_{ij}^{k} - F(w_i^* - w_j^*)\mathbf{1}_k)^{\mathrm{T}} A^{(k)} (y_{ij}^{k} - F(w_i^* - w_j^*)\mathbf{1}_k)}_{I_{ij}^{1,k}} + (F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))^2 \\
&\quad \underbrace{- 2(F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))\left(\frac{\bar{Y}_{ij}^{k}}{k} - F(w_i^* - w_j^*)\right)}_{I_{ij}^{2,k}},
\end{aligned}$$

where in ζ_1 we have $A^{(k)} \triangleq \frac{\mathbf{1}_k \mathbf{1}_k^{\mathrm{T}} - I_k}{k(k-1)}$ and $y_{ij}^k \in \mathbb{R}^k$ is a vector such that $y_{ij}^k = [Y_{ij}^1, \ldots, Y_{ij}^k]$ and in ζ_2 the term $I_{ij}^{1,k}$ follows by observing that

$$\left(y_{ij}^k - F(w_i^* - w_j^*)\mathbf{1}_k\right)^{\mathrm{T}} A^{(k)} \left(y_{ij}^k - F(w_i^* - w_j^*)\mathbf{1}_k\right) = (y_{ij}^k)^{\mathrm{T}} A^{(k)} y_{ij}^k + \mathbf{1}_k^{\mathrm{T}} A^{(k)} \mathbf{1}_k F(w_i^* - w_j^*)^2 - 2p_{ij} \mathbf{1}_k^{\mathrm{T}} A^{(k)} y_{ij}^k. \tag{45}$$

Now, we will upper bound the quadratic variation term $\sum_{(i,j)\in\mathcal{E}} I_{ij}^{1,k}$. For this we will utilize Lemma 5 to obtain the tail bounds for any $\nu \in (0, 1/e)$ to obtain (for some constant c):

$$\mathbb{P}\left(\exists k \geq 2 : \sum_{(i,j)\in\mathcal{E}} I_{ij}^{1,k} > c\frac{\sqrt{|\mathcal{E}|}}{k}\ell_{k,\nu}\right) \leq \frac{\nu}{2}.$$

It is straightforward to show that $\sum_{(i,j)\in\mathcal{E}} I_{ij}^{2,k}$ is $(4\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2/k)$-sub-gaussian. Therefore, by utilizing time-uniform version of Hoeffding inequality [48, Corollary 8] for the user-defined $h(k) = (\pi k)^2/6$ (also used for the stitching argument in the proof of Lemma 5), we obtain for any $\nu \in (0, 1)$

$$\mathbb{P}\left(\exists k \geq 2 : \sum_{(i,j)\in\mathcal{E}} I_{ij}^{2,k} > -2\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}\sqrt{2\frac{\log(h(\log_2(k))) + \log(2/\nu)}{k/2}}\right) \leq \frac{\nu}{2}.$$

Combining the two tail bounds and a simple calculation completes the proof for type I error.

Part 2. Observe that under hypothesis H_1 we have

$$\begin{aligned}
T_{ij}^k - (p_{ij} - F(w_i^* - w_j^*))^2 &= \frac{\bar{Y}_{ij}^k(\bar{Y}_{ij}^k - 1)}{k(k-1)} - p_{ij}^2 + F(\hat{w}_i - \hat{w}_j)^2 - F(w_i^* - w_j^*)^2 \\
&\quad + 2\left(F(w_i^* - w_j^*)p_{ij} - F(\hat{w}_i - \hat{w}_j)\frac{\bar{Y}_{ij}^k}{k}\right) \\
&\overset{\zeta_1}{=} (y_{ij}^k)^{\mathrm{T}} A^{(k)} y_{ij}^k - \mathbf{1}_k^{\mathrm{T}} A^{(k)} \mathbf{1}_k p_{ij}^2 + F(\hat{w}_i - \hat{w}_j)^2 - F(w_i^* - w_j^*)^2 \\
&\quad + 2p_{ij}(F(w_i^* - w_j^*) - F(\hat{w}_i - \hat{w}_j)) + 2\left(p_{ij} - \frac{\bar{Y}_{ij}^k}{k}\right)F(\hat{w}_i - \hat{w}_j),
\end{aligned} \tag{46}$$

where in ζ_1 we have $A^k \triangleq \frac{\mathbf{1}_k \mathbf{1}_k^{\mathrm{T}} - I_k}{k(k-1)}$ and $y_{ij}^k \in \mathbb{R}^k$ is avector $y_{ij}^k = [y_{ij}^\prime, \ldots, y_{ij}^k]$ as before. Now, observe that the term

$$\begin{aligned}
(y_{ij}^k)^{\mathrm{T}} A^{(k)} y_{ij}^k - \mathbf{1}_k^{\mathrm{T}} A^{(k)} \mathbf{1}_k p_{ij}^2 &= \left(y_{ij}^k - p_{ij}\mathbf{1}_k\right)^{\mathrm{T}} A^{(k)} \left(y_{ij}^k - p_{ij}\mathbf{1}_k\right) \\
&\quad + 2p_{ij}\left(y_{ij}^k - p_{ij}\mathbf{1}_k\right)^{\mathrm{T}} A^{(k)} \mathbf{1}_k \\
&= \left(y_{ij}^k - p_{ij}\mathbf{1}_k\right)^{\mathrm{T}} A^{(k)} \left(y_{ij}^k - p_{ij}\mathbf{1}_k\right) + 2p_{ij}\left(\frac{\bar{Y}_{ij}^k}{k} - p_{ij}\right).
\end{aligned}$$

Substituting the above bound in (46), we obtain

$$\begin{aligned}
T_{ij}^k - (p_{ij} - F(w_i^* - w_j^*))^2 &= \underbrace{\left(y_{ij}^k - p_{ij}\mathbf{1}_k\right)^{\mathrm{T}} A^{(k)} \left(y_{ij}^k - p_{ij}\mathbf{1}_k\right)}_{I_{ij}^{3,k}} \\
&\quad + \underbrace{2(p_{ij} - F(\hat{w}_i - \hat{w}_j))\left(\frac{\bar{Y}_{ij}^k}{k} - p_{ij}\right)}_{I_{ij}^{4,k}} \\
&\quad + \underbrace{(F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))(F(\hat{w}_i - \hat{w}_j) + F(w_i^* - w_j^*) - 2p_{ij})}_{I_{ij}^{5,k}}.
\end{aligned}$$

Now the term $\sum_{(i,j)\in\mathcal{E}} I_{ij}^{3,k}$ is bounded by utilizing Lemma 5 to obtain the tail bounds for some constant c

$$\mathbb{P}\left(\exists k \geq 2 : \sum_{(i,j)\in\mathcal{E}} I_{ij}^{3,k} > -c\frac{\sqrt{|\mathcal{E}|}}{k}\ell_{k,\nu}\right) \leq \frac{\nu}{2}.$$

And the term $\sum_{(i,j)\in\mathcal{E}} I_{ij}^{4,k}$ is bounded by utilizing adaptive Hoeffding's inequality [48, Corollary 8] to obtain the tail bounds

$$\mathbb{P}\left(\exists k \geq 2: \sum_{(i,j)\in\mathcal{E}} I_{ij}^{4,k} > -4\|P - \hat{\mathsf{F}}\|_{\mathrm{F}}\sqrt{\frac{\ell_{k,\nu}}{k}}\right) \leq \frac{\nu}{2}.$$

An application of triangle inequality to the above equation yields

$$\mathbb{P}\left(\exists k \geq 2: \sum_{(i,j)\in\mathcal{E}} I_{ij}^{4,k} > -4(\|P - \mathsf{F}\|_{\mathrm{F}} + \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}})\sqrt{\frac{\ell_{k,\nu}}{k}}\right) \leq \frac{\nu}{2}.$$

Finally, the term $\sum_{(i,j)\in\mathcal{E}} I_{ij}^{5,k}$ is bounded using Cauchy-Schwarz inequality as:

$$\begin{aligned}\sum_{(i,j)\in\mathcal{E}} I_{ij}^{5,k} &= \sum_{(i,j)\in\mathcal{E}} (F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))^2 \\ &\quad + \sum_{(i,j)\in\mathcal{E}} 2(F(\hat{w}_i - \hat{w}_j) - F(w_i^* - w_j^*))(F(w_i^* - w_j^*) - p_{ij}) \\ &\geq \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 - 2\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}\|\mathsf{F} - P\|_{\mathrm{F}}.\end{aligned}$$

Combining the three bounds for $I_{ij}^{3,k}, I_{ij}^{4,k}, I_{ij}^{5,k}$ completes the proof for type II error. □

Lemma 5 (Time-Uniform Quadratic Bound)**.** *For any element v in a finite set $\mathcal{V}$, consider a sequence of independent random variables $x_v^{(1)}, x_v^{(2)}, x_v^{(3)}, \ldots$ such that $x_v^{(l)} \sim \mathsf{Bernoulli}(p_v)$ for $l \in \mathbb{N}$ and some $p_v \in (0,1)$. Define the random vector $\bar{x}_v^{(k)} = \left(x_v^{(1)}, x_v^{(2)}, \ldots, x_v^{(k)}\right) \in \mathbb{R}^k$ and let $\{A^{(k)} \in \mathbb{R}^{k\times k} : k \in \mathbb{N}\}$ be a sequence of matrices such that $A^{(k)} = (\mathbf{1}_k\mathbf{1}_k^{\mathrm{T}} - I_k)/(k(k-1))$. Then, there exists a constant $c > 0$ such that for all $\nu \in (0, 1/e)$, we have*

$$\mathbb{P}\left(\exists k \geq 2, \sum_{v\in\mathcal{V}} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k)^{\mathrm{T}} A^{(k)} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k) > c\frac{\sqrt{|\mathcal{V}|}}{k}\log\left(3.5\log^2(k)/\nu\right)\right) \leq \nu.$$

Proof. Denote $s_v^{(k)} = (\bar{x}_v^{(k)} - p_v\mathbf{1}_k)^{\mathrm{T}} A^{(k)} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k)$. Recall that we had shown that $\{s_v^{(k)} : k \in \mathbb{N}\setminus 1\}$ forms a reverse-martingale with respect to canonical reverse filtration $\{\sigma\big(\sum_{m=1}^k x_v^{(m)}, x_v^{(k+1)}, x_v^{(k+2)}, \ldots\big) : k \in \mathbb{N}\setminus 1\}$, i.e., for $k \geq 2$, we have

$$\mathbb{E}\big[(\bar{x}_v^{(k)} - p_v\mathbf{1}_k)^{\mathrm{T}} A^{(k)} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k)|\mathcal{F}_{k+1}\big] = (\bar{x}_v^{(k+1)} - p_v\mathbf{1}_{k+1})^{\mathrm{T}} A^{(k+1)} (\bar{x}_v^{(k+1)} - p_v\mathbf{1}_{k+1}).$$

This fact follows from an expansion of the corresponding terms (similar to (45)) and then the proof follows as in Section VII-A. Now, for any $v \in \mathcal{V}$, we define a richer class of filtration known as exchangeable filtration $\{\tilde{\mathcal{F}}_k^v : k \in \mathbb{N}\setminus 1\}$ [53], which denotes the σ-algebra generated by all real-valued Borel-measurable functions of $x_v^{(1)}, x_v^{(2)}, x_v^{(3)}, \ldots$ which are permutation-symmetric in the first k arguments. It follows directly that $s_v^{(k)}$ is also a reverse-martingale with respect to $\tilde{\mathcal{F}}_k^v$. Therefore, by [48, Theorem 4], we have that the mapping $x \to \exp(\lambda x)$ for $\lambda \in (0,\infty)$, when applied to $s_v^{(k)}$, yields a reverse-submartingale with respect to filtration $\tilde{\mathcal{F}}_k^v$. Define the product σ-algebra $\tilde{\mathcal{F}}_k = \bigotimes_{v\in\mathcal{V}} \tilde{\mathcal{F}}_k^v$. Now, for any $k_0 \geq 2$ and $k_0 \in \mathbb{N}$, we have that

$$\begin{aligned}\mathbb{P}\left(\exists k \geq k_0 : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq u\right) &= \mathbb{P}\left(\exists k \geq k_0 : e^{\lambda\sum_{v\in\mathcal{V}} s_v^{(k)}} \geq e^{\lambda u}\right) \\ &\leq \frac{\mathbb{E}\Big[\exp\Big\{\big(\lambda\sum_{v\in\mathcal{V}} s_v^{(k_0)}\big)\Big\}\Big]}{e^{\lambda u}}.\end{aligned}$$

where the last step follows from Ville's inequality for nonnegative reverse submartingales [48, Theorem 2]. Also note that $s_v^{(k)} \leq 1$ for all k with probability 1, therefore $\mathbb{E}[e^{\lambda s_v^{(k)}}]$ always exists. Now note that

$$\begin{aligned}\sum_{v\in\mathcal{V}} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k)^{\mathrm{T}} A^{(k)} (\bar{x}_v^{(k)} - p_v\mathbf{1}_k) &= \sum_{v\in\mathcal{V}} \sum_{i,k=1 i\neq j}^{k} \frac{(x_v^{(i)} - p_v)(x_v^{(j)} - p_v)}{k(k-1)} \\ &= (\bar{x}_{\mathcal{V}}^{(k)})^{\mathrm{T}} A_{\mathcal{V}}^{(k)} (\bar{x}_{\mathcal{V}}^{(k)}),\end{aligned}$$

where $(\bar{x}_{\mathcal{V}}^{(k)})^{\mathrm{T}} = [(\bar{x}_{v_1}^{(k)} - \mathbf{1}_k p_{v_1})^{\mathrm{T}}, \dots, (\bar{x}_{v_{|\mathcal{V}|}}^{(k)} - \mathbf{1}_k p_{v_{|\mathcal{V}|}})^{\mathrm{T}}]$ is a vector in $\mathbb{R}^{k|\mathcal{V}|}$ formed by concatenating the vectors $\bar{x}_{v_i}^{(k)}$ for all $v_i \in \mathcal{V}$ and $i \in [|\mathcal{V}|]$. And $A_{\mathcal{V}}^{(k)} \in \mathbb{R}^{|\mathcal{V}|k \times |\mathcal{V}|k}$ formed by stacking the matrices $A_v^{(k)}$ as a diagonal block structure. Now by [50, Theorem 1.1], we have that there exists constants c, c' such that we have

$$\mathbb{E}\Big[e^{\lambda \sum_{v\in\mathcal{V}} s_v^{(k_0)}}\Big] \leq \exp\Big(-c\lambda^2 \|A_{\mathcal{V}}^{(k_0)}\|_{\mathrm{F}}^2\Big) \text{ for } \lambda \leq c'/\|A_{\mathcal{V}}^{(k_0)}\|_2,$$

where $\|\cdot\|_2$ denotes the spectral norm. Therefore, we have that

$$\mathbb{P}\Bigg(\exists k \geq k_0 : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq u\Bigg) \leq \exp\Big(-\lambda u + c\lambda^2 \|A_{\mathcal{V}}^{(k_0)}\|_{\mathrm{F}}^2\Big) \text{ for } \lambda \leq c'/\|A_{\mathcal{V}}^{(k_0)}\|_2.$$

Now, by optimizing over λ, we can conclude that

$$\mathbb{P}\Bigg(\exists k \geq k_0 : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq u\Bigg) \leq \exp\Bigg(-c\min\Bigg\{\frac{u^2}{\|A_{\mathcal{V}}^{(k_0)}\|_{\mathrm{F}}^2}, \frac{u}{\|A_{\mathcal{V}}^{(k_0)}\|_2}\Bigg\}\Bigg). \tag{47}$$

By the block structure of $A_{\mathcal{V}}^{(k_0)}$, we have the following

$$\|A_{\mathcal{V}}^{(k_0)}\|_{\mathrm{F}}^2 = \frac{|\mathcal{V}|}{k_0(k_0-1)}, \quad \text{and} \quad \|A_{\mathcal{V}}^{(k_0)}\|_2 = \max_{v\in\mathcal{V}} \|A_v^{(k_0)}\|_2 = \frac{1}{k_0}.$$

Substituting, the above values in (47) we obtain that for all $\nu \in (0, 1/e)$, we have the following result for some constant $\tilde{c}$:

$$\mathbb{P}\Bigg(\exists k \geq k_0 : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq \tilde{c}\frac{\sqrt{|\mathcal{V}|}\log(1/\nu)}{k_0}\Bigg) \leq \nu$$

The rest of the proof follows by a stitching argument [54] and is provided below for completeness. For any $\nu \in (0, 1/e)$, define a function $h(k) = \frac{(\pi k)^2}{6}$. Now, observe that

$$\begin{aligned}
&\mathbb{P}\Bigg(\exists k \geq 2 : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq \tilde{c}\frac{\sqrt{|\mathcal{V}|}}{k}\log\big(\frac{h(\log_2 k)}{\nu}\big)\Bigg) \\
&\qquad\qquad \leq \sum_{l=1}^{\infty} \mathbb{P}\Bigg(\exists k \geq 2^l : \sum_{v\in\mathcal{V}} s_v^{(k)} \geq \tilde{c}\frac{\sqrt{|\mathcal{V}|}}{2^l}\log\big(\frac{h(l)}{\nu}\big)\Bigg) \\
&\qquad\qquad \leq \sum_{l=1}^{\infty} \frac{\nu}{h(l)} = \nu.
\end{aligned}$$

Finally, the statement in the lemma follows by a simple calculation. □

C. Proof of Theorem 5

Fix $t \geq 1$, and recall that from Lemma 2, for some constant c_0 we have that with probability atmost e^{-t}, $\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}} \geq \sqrt{c_0 tn/k_1}$. This implies that with probability at least $1 - e^{-t}$, we have

$$\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 + c\frac{\sqrt{|\mathcal{E}|}}{k}\ell_{k,\nu} + 4\frac{\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}}{\sqrt{k}}\sqrt{\ell_{k,\nu}} \leq \frac{c_0 tn}{k_1} + c\frac{|\mathcal{E}|^{\frac{1}{2}}}{k}\ell_{k,\nu} + 4\sqrt{\frac{c_0 tn\ell_{k,\nu}}{kk_1}}.$$

Therefore, by using a basic union-bound argument to the bound under null hypothesis in Lemma 4, we have

$$\mathbb{P}_{H_0}\Bigg(\exists k \geq 2, T^{k_1,k} \geq \frac{c_0 tn}{k_1} + c\frac{|\mathcal{E}|^{\frac{1}{2}}}{k}\ell_{k,\nu} + 4\sqrt{\frac{c_0 tn\ell_{k,\nu}}{kk_1}}\Bigg) \leq \nu + e^{-t}.$$

The bound on type II error follows by a similar argument. First, using basic algebra, and using D to denote $\|P - \mathsf{F}\|_{\mathrm{F}}$, we obtain from the bound under H_1 in Lemma 4 that:

$$\mathbb{P}_{H_1}\Bigg(\exists k \geq 2, T^{k_1,k} \geq (D - \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}})^2 - c\frac{|\mathcal{E}|^{\frac{1}{2}}}{k}\ell_{k,\nu} - 4\frac{(D + \|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}})}{\sqrt{k}}\sqrt{\ell_{k,\nu}}\Bigg) \leq \nu.$$

Now utilizing the fact that the $D \geq \sqrt{c_0 tn/(k_1)}$ and the same union bound technique as above, we obtain that

$$\mathbb{P}_{H_1}\Bigg(\exists k \geq 2,\ T^{k_1,k} - \Bigg(D - \sqrt{\frac{c_0 tn}{k_1}}\Bigg)^2 \leq -\frac{c|\mathcal{E}|^{\frac{1}{2}}\ell_{k,\nu}}{k}$$

$$-4\left(\frac{D}{\sqrt{k}}+\sqrt{\frac{c_0tn}{k_1k}}\right)\sqrt{\ell_{k,\nu}}\right) \leq \nu + e^{-t}.$$

Since the above bounds for any pairwise comparison matrix P satisfy Assumption 1 and $n\epsilon = \Theta(D)$ by Theorem 1, we can take supremum with respect to P in classes $\mathcal{M}_0$ and $\mathcal{M}_1(\sqrt{\tilde{c}_0 t/(nk_1)})$ on type I and type II error probability bounds, respectively. This completes the proof of the theorem. $\square$

VIII. Experiments

In this section, we develop a data-driven approach to select the threshold for our test T, and conduct simulations to validate our theoretical results and its evaluations on synthetic and real-world datasets.

Estimating the threshold. Given a pairwise comparison dataset $\mathcal{Z} \triangleq \{Z_{ij}^m : (i,j) \in \mathcal{E},\ m \in [k_{i,j}]\}$, we employ an empirical quantile approach to determine the critical threshold for our hypothesis testing problem. We generate multiple $\mathcal{T}_F$ models with random skill scores $w \in \mathbb{R}^n$ such that $\|w\|_\infty \leq b$ for some constant b and simulate k_{ij} "i vs. j" comparisons by sampling binomial random variables $\{\tilde{Z}_{ij} \sim \text{Bin}(k_{ij}, F(w_i - w_j)\}_{(i,j)\in\mathcal{E}}$. We then compute the test statistic T for each simulated dataset, repeating the process a sufficient number of times to build a distribution of test statistics. Finally, we extract the 95th percentile value (or 0.95 quantile) from this distribution as our empirical threshold.

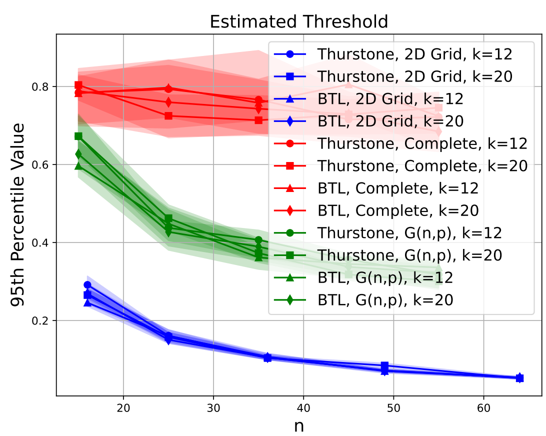

Fig. 1: Estimated scaled threshold for various values of n, k, graph topologies, and $\mathcal{T}_F$ models.

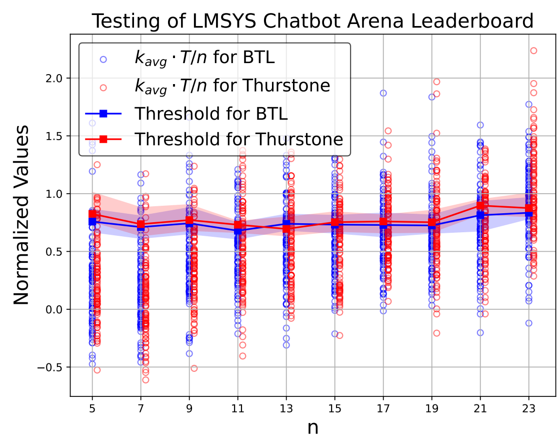

Fig. 2: Scaled test statistics and the estimated thresholds evaluated on the LYMSYS dataset.

In our first experiment, we investigate the behaviour of threshold for test T based on this empirical quantile approach for various values of n and k, and for different graph topologies and $\mathcal{T}_F$ models. We considered values of n ranging from 15 to 55 with intervals of 10, $k \in \{12, 20\}$, and graph topologies including complete graphs, $\lceil\sqrt{n}\,\rceil \times \lceil\sqrt{n}\,\rceil$ toroidal grids and sparse graphs generated from Erdős-Rényi $\mathcal{G}(n,p)$ model with parameter $p = 2\log^2(n)/n$ and the $\mathcal{T}_F$ models included standard Thurstone (Case V) and BTL models. For each choice of parameters, we generated 400 models by randomly sampling weights and with $b = F^{-1}(0.98)/2$ and generated synthetic comparison data. The scaled test-statistic $k\lambda_2(L)(\mathcal{G}) \cdot T/(nd_{\max})$ was computed for every parameter choice and the 95th percentile value of this scaled T was identified as the threshold γ. Fig. 1 plots this 95th percentile values with respect to n for various parameter choices. Notably, the value γ remains roughly constant with n, k and model F for complete graphs. However, for toroidal grid graphs, a significant decrease in the scaled test statistic is observed, suggesting that our error bounds may be overly conservative in estimating the deviation in T.

In our next experiment, we apply our test to the LMSYS chatbot leaderboard [55], a widely used benchmark for evaluating the performance of LLMs. The dataset contains a collection of pairwise comparisons between various LLMs based on their response to prompts, which are then used to obtain ELO ratings. We retain the directional nature of comparisons, where an "i vs. j" comparison indicates model i as the first response and j as second during the evaluation. We rank the LLMs based on their frequency of appearance in the dataset and perform the test repeatedly on top n LLMs in this ordering, with n ranging from 5 to 21 with gaps of 2, for both Thurstone and BTL models. For each n, we plot the values in Fig. 2 of (scaled) test-statistic $k_{\text{avg}} \cdot T/n$ and the obtained (scaled) thresholds using the quantile approach (with same parameters as above), where k_{avg} is the average of k_{ij} over all $(i,j) \in \mathcal{E}$. By randomizing over the partitioning of dataset $\mathcal{Z}$ into $\mathcal{Z}_1$ and $\mathcal{Z}_2$ and computing T each time, we essentially obtain a distribution of T and plot these values in Fig. 2 as a scatter plot. The figure highlights that both BTL and Thurstone models perform well in modeling for smaller values of n with only 10% of samples above the threshold but exhibit significant deviations for larger values of n (as around 60% samples are above the threshold for $n = 21$). Notably, for both experiments, the error bars are 96% confidence intervals (see Appendix B for additional details), and all results were obtained using modest computational resources within a few minutes to an hour.

IX. Conclusion

In this work, we developed a rigorous hypothesis testing framework to determine whether pairwise comparison data is generated by an underlying generalized Thurstone model with a given choice function F. Our analysis introduced the notion of separation distance to quantify the deviation of a pairwise comparison model from the set of $\mathcal{T}_F$ models, enabling us to frame the hypothesis testing problem in a minimax sense. Leveraging this formulation, we derived both upper and lower bounds on the critical threshold of our testing problem, revealing key dependencies on the topology of the observation graph. These bounds were shown to be tight for certain graph classes, such as complete graphs.

In addition, we proposed a hypothesis test based on the separation distance and established theoretical guarantees, including "time-uniform" bounds on Type I and Type II error probabilities, as well as a minimax lower bound on the risk of the testing problem. Alongside this, auxiliary results such as error bounds for parameter estimation and confidence intervals under the null hypothesis were derived. To validate our findings, we conducted experiments on both synthetic and real-world datasets and introduced a data-driven approach for determining the test threshold.

This study opens up several avenues for future research. For instance, extending the hypothesis testing framework to handle general multi-way comparisons rather than pairwise comparisons. Other directions include developing active testing techniques within the framework of generalized Thurstone models that optimize test performance. Finally, an important avenue is extending the testing and estimation framework to dependent data, as real-world data often exhibits correlations that could affect estimation as well as testing results.

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Appendix A
Simplified Expressions for Type I and II Error Probabilities

We obtain the following corollaries by plugging in the parameter values in Theorem 5 for a complete graph and single cycle on n nodes.

Corollary 1 (Type I and Type II Error Probability Bounds for Complete Graph)**.** *For the setting in Theorem 5 assume that we have a complete graph on n nodes, then there exists (different) constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that for $\epsilon \geq c_5/\sqrt{nk_1}$, such that for all $\nu \in (0, 1/e)$ and $t \geq 1$, we have*

$$\mathbb{P}_{H_0}\left(\exists k \geq 2, T^{k_1,k} \geq \frac{c_1 tn}{k_1} + \frac{c_2 n\ell_{k,\nu}}{k} + \sqrt{\frac{nt\ell_{k,\nu}}{kk_1}}\right) \leq \nu + e^{-t},$$

$$\mathbb{P}_{H_1}\left(\exists k \geq 2, T^{k_1,k} \geq \left(D - c_3\sqrt{\frac{tn}{k_1}}\right)^2 - \frac{c_2 n \ell_{k,\nu}}{k} - \left(4\frac{D}{\sqrt{k}} + c_4\sqrt{\frac{tn}{kk_1}}\right)\sqrt{\ell_{k,\nu}}\right) \leq \nu + e^{-t}.$$

Corollary 2 (Type I and Type II Error Probability Bounds for Single Cycle Graph)**.** *For the setting in Theorem 5, assume that we have a single cycle graph on n nodes, then there exists (different) constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that for $\epsilon \geq c_5/\sqrt{nk_1}$, such that for all $\nu \in (0, 1/e)$ and $t \geq 1$, we have*

$$\mathbb{P}_{H_0}\left(\exists k \geq 2, T^{k_1,k} \geq \frac{c_1 tn}{k_1} + \frac{c_2\sqrt{n}\ell_{k,\nu}}{k} + \sqrt{\frac{nt\ell_{k,\nu}}{kk_1}}\right) \leq \nu + e^{-t},$$

$$\mathbb{P}_{H_1}\left(\exists k \geq 2, T^{k_1,k} \geq \left(D - c_3\sqrt{\frac{tn}{k_1}}\right)^2 - \frac{c_2\sqrt{n}\ell_{k,\nu}}{k} - \left(4\frac{D}{\sqrt{k}} + c_4\sqrt{\frac{tn}{kk_1}}\right)\sqrt{\ell_{k,\nu}}\right) \leq \nu + e^{-t}.$$

Appendix B
Experiments Details and Additional Experiments

In this appendix, we will provide additional details about the experiments that were performed in Section VIII, and in addition, we will empirically calculate the confidence intervals based on the expression in Proposition 4.

A. Additional Details for Experiments

In this section, we provide additional details on the experimental setup and methodology for the experiments in Section VIII.

Error bars and estimation of $\hat{w}$. To estimate the 95th quantile of the test statistics T, we used 400 samples. The error bars were based on the two-sided distribution-free conservative estimates presented in [56]. Specifically, for the 95th quantile, the upper 96% confidence interval was computed as the 97th quantile of the computed tests T, and the lower confidence interval was computed as the 92.5% quantile. Moreover, in all of our experiments the estimation of $\hat{w}$ was performed using standard gradient descent algorithm with a learning rate of 0.01 and for a maximum of 3000 iterations, until the norm of the gradient was less than 10^{-5}.

Testing on LYMSYS dataset. In our experiment on LYMSYS dataset, we used a maximum of 200 samples per pair and discarded pairs with fewer than 30 observed comparisons to reduce the imbalance in the data across pairs. The observation graph was a complete graph except for the larger values of n which had a few edges missing.

B. Confidence Intervals Under Null Hypothesis

In this sub-section, we will discuss a method to approximately calculate the confidence intervals under the null hypothesis, with a focus on the BTL model. While our discussion is specific to the BTL model, it can be easily generalized to other Thurstone models. Specifically, our goal is to approximately calculate the constants in Proposition 4 and as well as approximate the distribution of $\|\hat{\mathsf{F}} - \mathsf{F}\|_{\mathrm{F}}$. For the former, we will estimating the constants by conducting some simulation while for the latter, we will be utilizing gaussian approximation based on the asymptotic normality of $\hat{w} - w^*$ which was proved for the BTL model [35, Proposition 4.1]. Similar results have been established for the general Thurstone models as in [36].

Estimating constant c_7 in Proposition 4. To estimate the constant, we plot several trajectories of the normalized stochastic process $\frac{\sqrt{k}}{|\mathcal{V}|}\sum_{v\in\mathcal{V}}(\bar{x}_v^{(k)} - p_v\mathbf{1}_k)^{\mathrm{T}}A^{(k)}(\bar{x}_v^{(k)} - p_v\mathbf{1}_k)$ with $\bar{x}_v^{(k)}$ generated as in Lemma 5 and the p_v selected uniformly at random from $(0, 1)$. The values of $|\mathcal{V}|$ varied from 10 to 100 with gaps of 10. Fig. 3 plots the various trajectories of the (normalized) stochastic process as a function of k and also plots the 95th quantile for the stochastic process for all $k \in [100]$. The figure suggests that $c_7 \approx 0.45$ is a good approximation to the value of c_7 (in the fixed sample setting).

Estimating the quantile of $\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}$. In order to estimate $\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}$, we will utilize the asymptotic normality of vector $\Delta = \hat{w} - w^*$. Let $\hat{w}$ be computed as in (7), then under the conditions of mild regularity conditions it was shown in [35, Proposition 4.1] that

$$(\rho_1(\hat{w})(\hat{w}_1 - w_1^*), \ldots, \rho_n(\hat{w})(\hat{w}_n - \hat{w}_n^*)) \xrightarrow{d} \mathcal{N}(0, I_k),$$

where $\rho_i(w) = \sqrt{k\sum_{j:(i,j)\in\mathcal{E}} F'(\hat{w}_i - \hat{w}_j)}$ and $\xrightarrow{d}$ denotes the convergence in distribution. We will utilize this asymptotic normality result to approximate the distribution of $\|\hat{\mathsf{F}} - \mathsf{F}\|_{\mathrm{F}}$ using the delta method as:

$$\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2 = \sum_{(i,j)\in\mathcal{E}} \left(F(w_i^* - w_j^*) - F(\hat{w}_i - \hat{w}_j)\right)^2$$

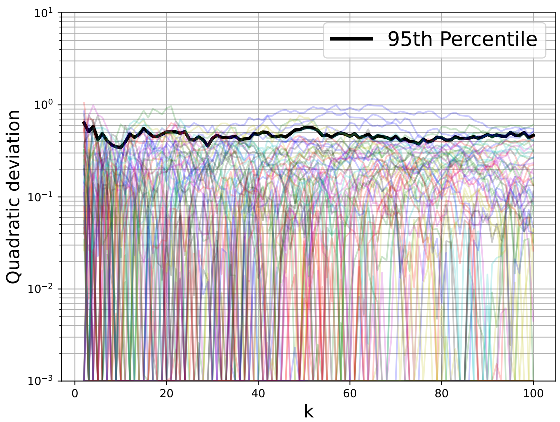

Fig. 3: Plot of various trajectories of stochastic process in Lemma 5.

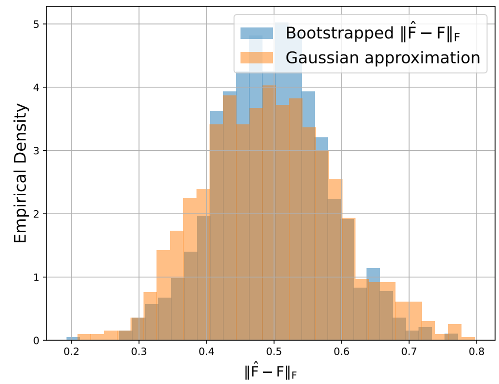

Fig. 4: Histogram of $\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2$ based on bootstrapping vs. asymptotic approximation.

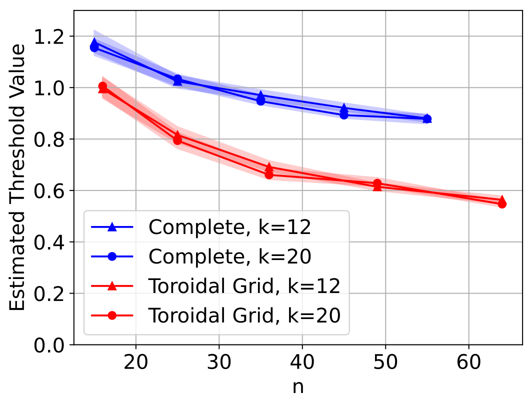

Fig. 5: Estimated threshold based on Proposition 4.

$$\approx \sum_{(i,j)\in\mathcal{E}} F'(\hat{w}_i - \hat{w}_j)^2 \big((w_i^* - \hat{w}_i) - (w_j^* - \hat{w}_j)\big)^2$$
$$= \sum_{(i,j)} F'(\hat{w}_i - \hat{w}_j)^2 (\Delta_i - \Delta j)^2 = 2\Delta^{\mathrm{T}} L_F(\hat{w})\Delta,$$

where we define $L_F(w)$ to be the following matrix

$$(L_F(w))_{ij} = \begin{cases} -F'(w_i - w_j)^2 & \text{if } (i,j)\in\mathcal{E} \\ \sum_{j:(i,j)\in\mathcal{E}} F'(w_i - w_j)^2 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Since Δ is asymptotically normal (as $k \to \infty$), therefore we approximate the distribution of $\|\mathsf{F} - \hat{\mathsf{F}}\|_{\mathrm{F}}^2$ with distribution of $2\Delta^{\mathrm{T}}(\hat{w})L_F(\hat{w})\Delta(\hat{w})$ where $\Delta_i(w) \sim \mathcal{N}\Big(0, \frac{1}{k\sum_{j:(i,j)\in\mathcal{E}} F'(w_i - w_j)}\Big)$. In Fig. 4, we plot the empirical distribution of $\|F(\hat{w}) - F(w^*)\|_F$ calculated by randomizing over the choice of partitioning of $\mathcal{Z}$ into $\mathcal{Z}_1$ and $\mathcal{Z}_2$. We also plot its asymptotic approximation, i.e., the empirical distribution of $2\Delta^{\mathrm{T}}(\hat{w})L_F(\hat{w})\Delta(\hat{w})$. Clearly, as can be seen in Fig. 4, our asymptotic approximation does indeed well approximate the empirical distribution even for a small number of samples. Finally, based on the the 0.95-quantile of the empirical distribution of $2\Delta^{\mathrm{T}}(\hat{w})L_F(\hat{w})\Delta(\hat{w})$, we compute the expression in for $c_7 = 0.45$ and plot the estimated confidence intervals in Fig. 5. It can be observed that for a complete graph, our estimated threshold values indeed do approach the value of 0.8 in Fig. 2 computed via empirical quantile method.